THE ROYAL SCHOOL OF SIGNALS

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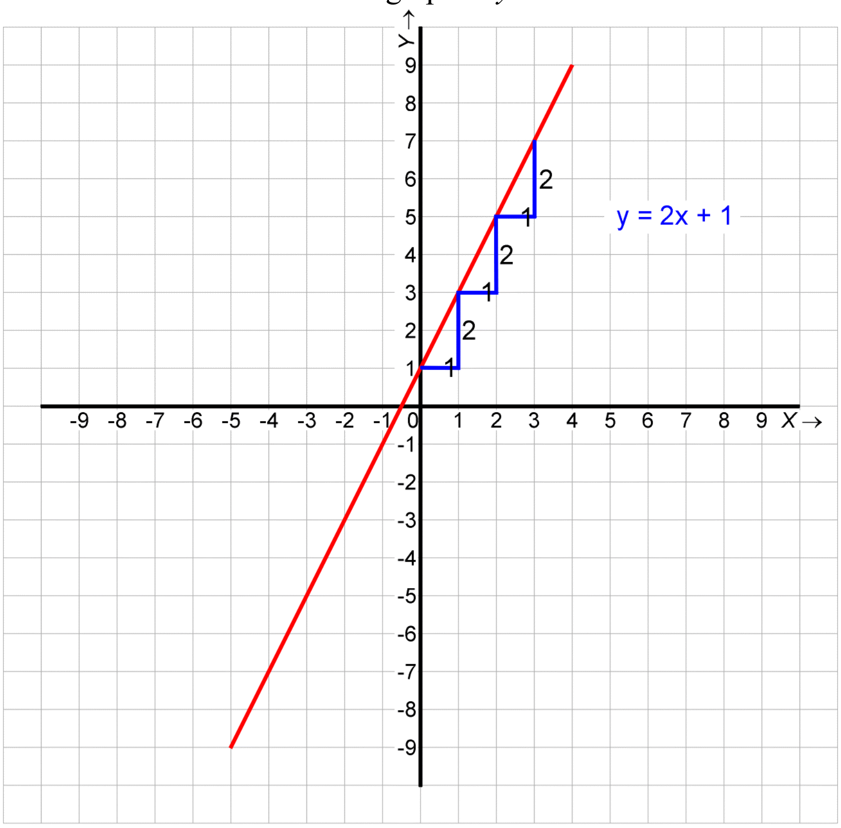
C O N T E N T S

Chapter 1

Gradients of lines and curves

Graph of a straight line

Consider the graph $y = 2x + 1$



p362 fig1

We can see that as x increases by 1 unit, y always increases by 2 units. This is the same at every point on the line.

Gradient

The ratio $\dfrac{\text{Increase in } y}{\text{Increase in } x}$

is called the *gradient* or *slope* of the line.

We can liken it to the gradient of a hill. The greater the gradient, the steeper the hill.

In this case the gradient $= 2$, which is the coefficient of x.

It is clear that the gradient of a straight line is the same anywhere on the line and is always given by the coefficient of x, since the coefficient of x determines how many units y increases for each unit increase in x.

To measure the gradient of a straight line we can take this ratio anywhere along the line, for example if we measure the changes in y and x along the whole length as shown in the diagram, it can be seen that:

total increase in y = 18 units
total increase in x = 9 units

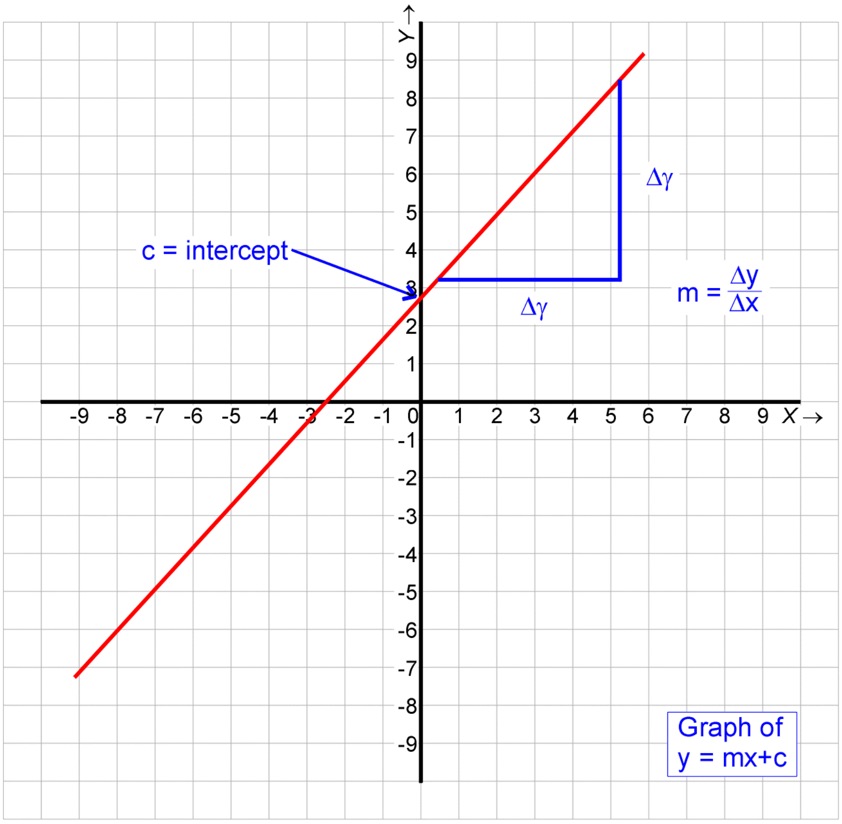
$\therefore$ gradient $= 18/9 = 2$.

In practice, if we wish to measure the gradient of a line, measuring over the longest possible section gives the most accurate result.

The equation $y = 2x + 1$ is called a linear equation since its graph is a straight line.

Intercept

If we let $x = 0$, we get $y = 1$. Therefore the line must intercept the y axis (which is the line $x = 0$) at the point where $y = 1$.



p362 fig2

Instead of writing "an increase in y" we write, for short, Δy or δy.
This is read as "delta Y". Similarly an increase in x is written as Δx.

Hence, the gradient of a straight line at any point is $\dfrac{\Delta y}{\Delta x}$

General form of the equation of a straight line

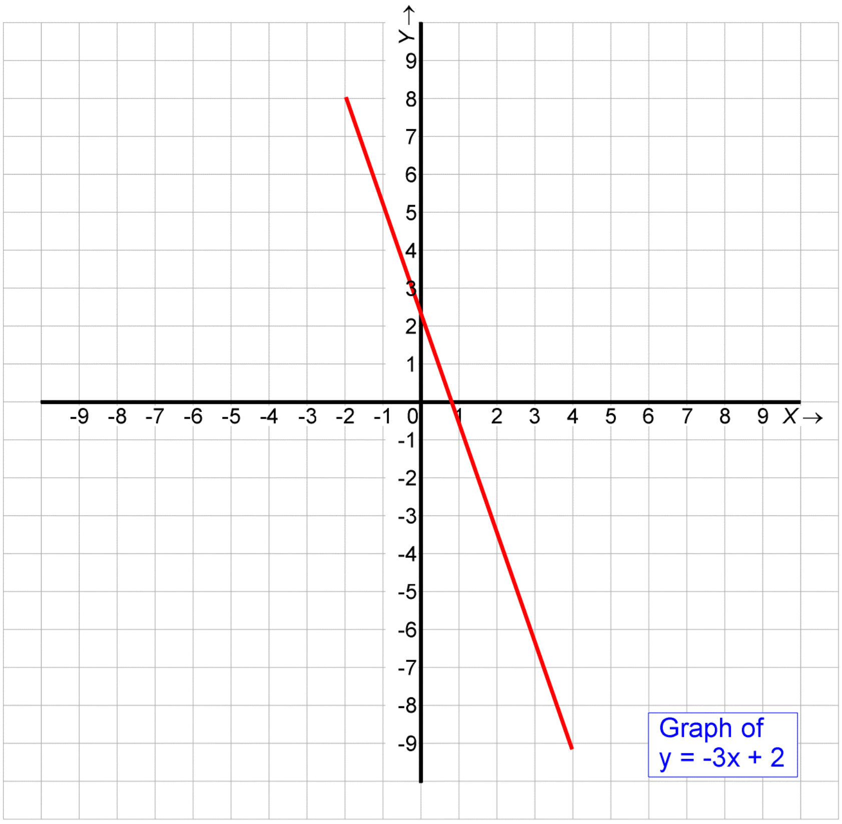
The equation of a straight line is of the form:

$$y = mx + c$$

m, the coefficient of x, is the gradient, since y increases proportionately by this amount with respect to x.
If we let $x = 0$, we get $y =$ c. Therefore c must be the intercept on the y axis.

Negative slope

If ${}^{\Delta y}/_{\Delta x}$ is negative this means that y is changing in a negative direction as x increases in the positive direction.
Consider the graph $y = -3x + 2$. As x increases by 1, y changes by –3.
Therefore ${}^{\Delta y}/_{\Delta x}$ is negative and equal to –3.
The intercept on the y axis is +2 as shown on the diagram below.



p362 fig3

SAQ3-1-1

Write down (a) the gradient (b) the intercept on the y axis of the following straight lines.

Equation	Gradient	Intercept
$y = 5x + 1$		
$y = -2x + 4$		
$Y = -1{\cdot}5x - 2$		
$y = x - 3$		
$y = 4$		

Plotting straight lines

There are 2 methods of drawing straight lines from an equation.

Method 1. Given a line $y = mc + c$, plot the intercept c on the y axis. from this point move along a distance x squares and then up or down a distance of mx squares.

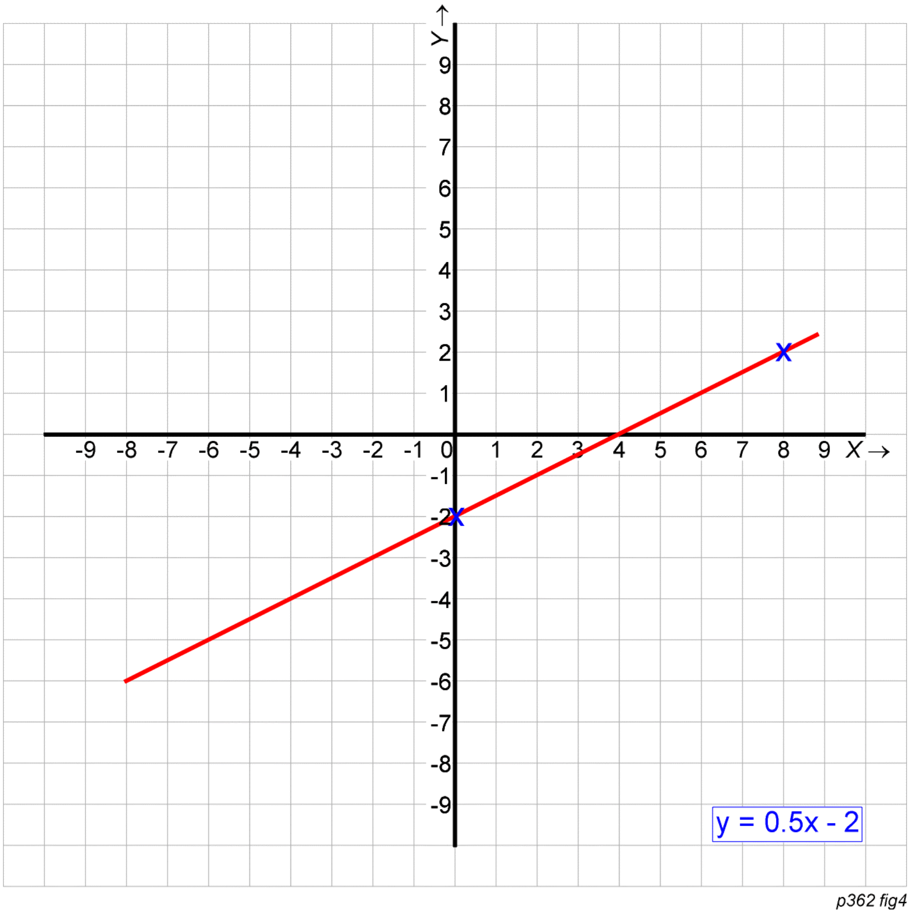
Method 2. Substitute values of x into the equation and calculate corresponding values of y. A straight line is determined uniquely by 2 points, but to draw it accurately it is better to plot 3 or more points to align your ruler correctly.

Example

Plot the graph $y = 0{\cdot}5x – 2$ for values of x between 8 and 8.

Substituting $x = –8$, $x = 0$, and $x = 8$, we get the corresponding value of $y = –6$, $y = –2$, $y = 2$.

Plotting the 3 points (–8, –6), (0, –2), (8, 2) and joining them, we obtain the line.



p362 fig4

Measuring the slope, ${}^{\Delta y}/_{\Delta x}$ we note that it is 0·5 as expected.

SAQ3-1-2

Plot the following lines on the same axes below.

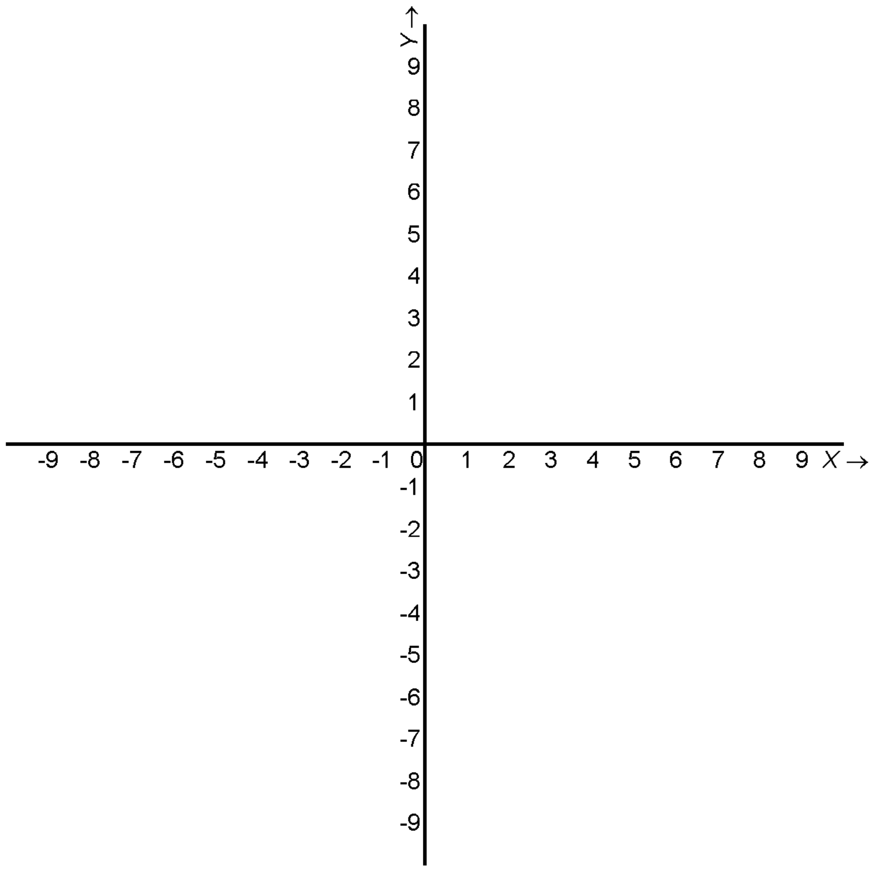
Measure the slope of each line and check that it is equal to the coefficient of x.

a. $y = 2x - 1$

b. $y = -2x + 2$

c. $y = 3x$

d. $y = -8$



p362 fig5

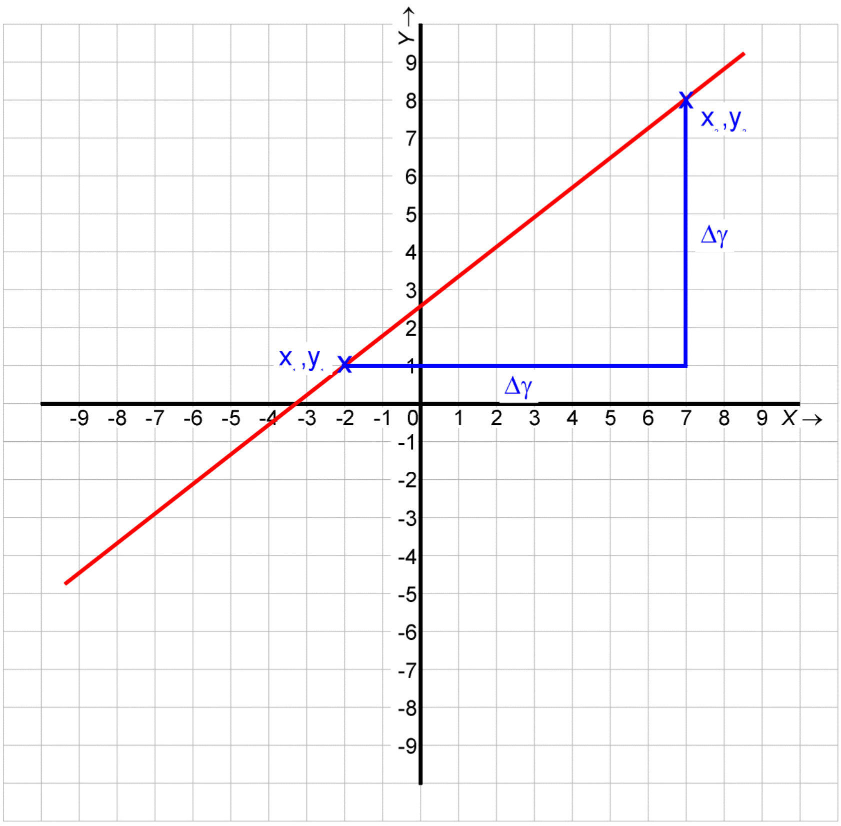
Equation of a line trough two points

Given the coordinates of 2 points, we can find the equation of the line through them.

Coordinates are conventionally written with the x coordinate first, e.g. the point (3, –2) means the point whose coordinates are $x = 3$, $y = –2$.

Consider the line, $y = mx + c$ through the 2 points (x_1, y_1) and (x_1, y_2)

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$



p362 fig6

Having found the gradient m, the intercept c may be found by substituting either coordinate pair into the equation.

Find the equation of the line through the points (–2, 1) and (6, 13).

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{13 - 1}{6 - -2} = \frac{12}{8} = 1 \cdot 5$$

$$\therefore \; y = 1{\cdot}5x + c$$

Substituting $x = –2$, $y = 1$ gives $1 = 1{\cdot}5 \times –2 + c$

$\therefore c = 4$

Hence the equation of the line is $y = 1{\cdot}5x + 4$

SAQ3-1-3

Find the equations of the straight lines through the following sets of points:

a. (1, 6) and (3, 20)

b. (–5, –2) and (4, 25)

c. (–2, 15) and (3, 5)

d. (1, 20) and (5, 4)

e. (–2, –4) and (6, 12)

Equations of straight lines in other forms

It is clear that the equation of a line could also be written in terms of x.

For example, the equation $y - 0{\cdot}5x + 3$ could equally well be written as

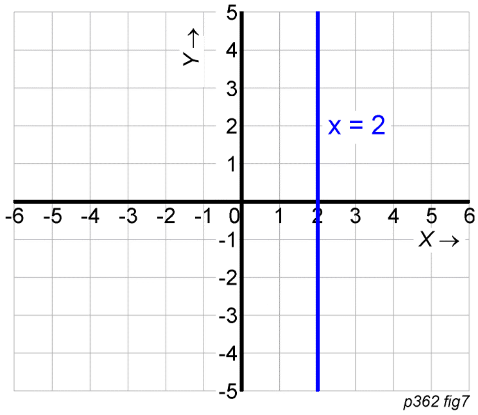
$$x = 2y - 6$$

There are occasions where it may be more convenient to express the equation in this way.

Another form of the equation is the implicit form ie $x - 2y + 6 = 0$

A straight line which is parallel to the y axis cannot be written in the form $y = mx + c$, since the gradient is infinite, but can only be written in terms of x. For example, the line $x = 2$.

The y axis is the line $x = 0$.



p362 fig7

On some graphs, instead of labelling the axes $x\rightarrow$ *and* $y\uparrow$, the y axis is labelled as the line $x = 0$ and the x axis as the line $y = 0$.

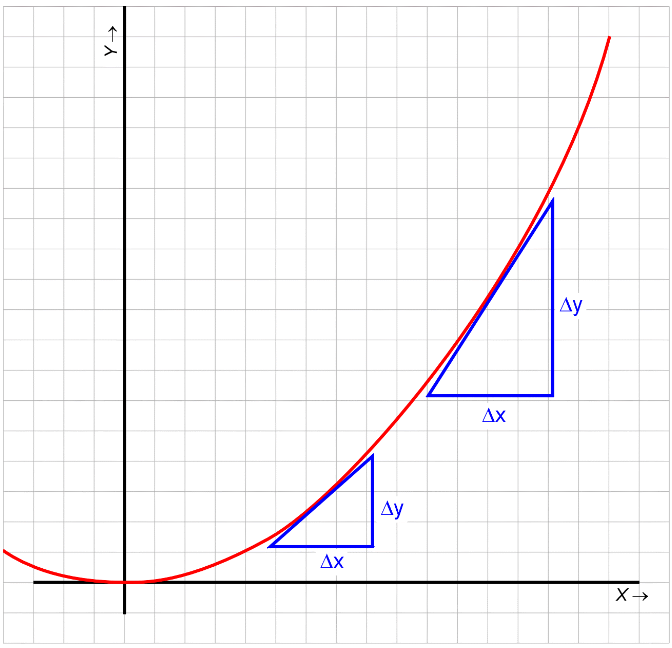
Gradient of a curve

The gradient of a curve is not the same at every point, so how do we define it?

The gradient of a curve at some point is defined as the gradient of the *tangent* to the curve at that point. A tangent is a line which touches a curve at one point only.

The gradient $^{\Delta y}/_{\Delta x}$ varies with x and therefore must be a function of x.

In practice, it is very difficult to draw a tangent to a curve.



p362 fig8

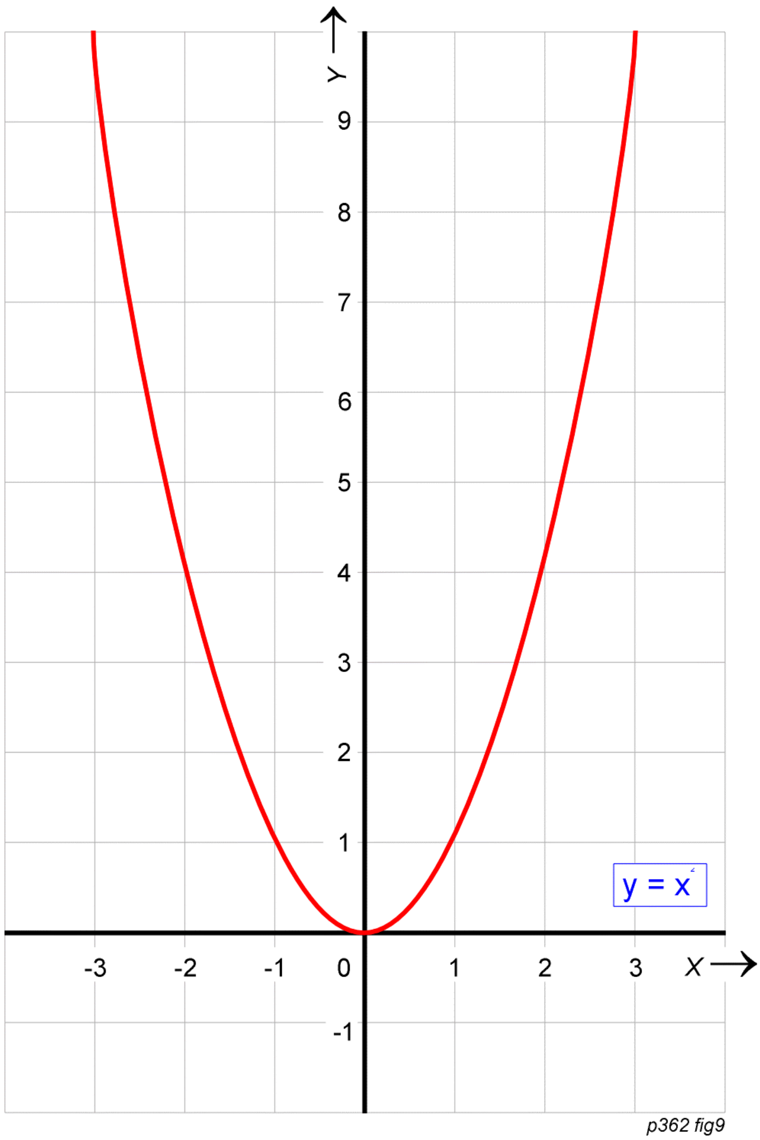
SAQ3-1-4

On the graph of $y = x^2$, (see next page) draw the tangents to the curve at

a. $x = 1$

b. $x = 0{\cdot}5$

c. $x = 2$



Measure as accurately as possible the gradient of the tangents at these points.

SAQ3-1-5

On the graph of sin x below, draw tangents and measure as accurately as possible the gradient at the points

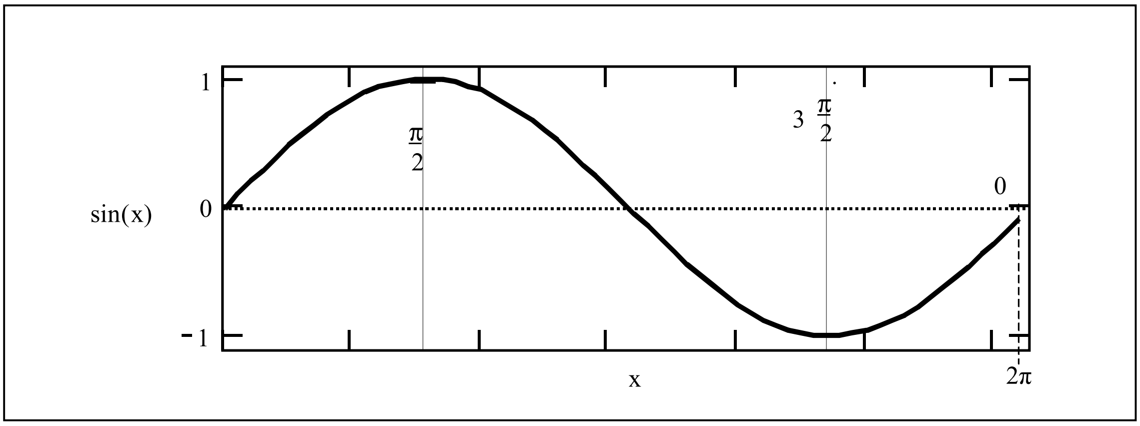
a. $x = 0$

b. $x = \pi$

c. $x = 2\pi$

Note that the same scale has been used on both x and y axes.

If different scales are used on the axes, which is often the case, the gradient measurement must be scaled accordingly.



Graph of *y* = sin *x* . Note : *x is in radians.*

What is the gradient at the points where $x = \pi/2$ and $x = 3\pi/2$?

Chapter 2

Elementary differentiation

Differentiation

Differential calculus allows us to find the rate of change of one variable with respect to another. We shall commonly use the letters y and x to denote variables but other letters are often used, particularly in practical problems.

Function notation

If y is a *function* of x this means that y varies with x according to some formula. y is called the dependent variable and x is called the independent variable.

We write $y = \mathrm{f}(x)$ meaning "y is a function of x". For example $y = x^2$, $y = \sin x$, $y = e^x$. These are all functions of the variable x. Alternatively, we may write $\mathrm{f}(x) = x^2$ instead of $y = x^2$.

Similarly, in electrical problems we may write $i = \mathrm{f}(t)$, where i is current and t is time. This implies that current is varying with time according to some relationship. i is called the *instantaneous* value of current since it is the value of current at some instant, t seconds.

$\mathrm{f}(a)$, where a is some number, means the function evaluated at $x = a$.
For example: if $\mathrm{f}(x) = x^2$, then $\mathrm{f}(3) = 9$
if $\mathrm{f}(x) = \sin x$, then $\mathrm{f}(\pi/2) = 1$

Rate of change

A **graph** is a pictorial representation of a function. The type of graph which we have used in this section plots y against x on axes at right–angles. This is called a Cartesian graph. Other types of graphs such as polar plots have specific applications.

The rate of change of y as x varies, is represented pictorially by the gradient of the graph. We have seen that the gradient of a straight line is a constant.
If $y = \mathrm{m}x + \mathrm{c}$ then y varies at the constant rate, m.

If $\mathrm{f}(x)$ is not a linear function, i.e. its graph is not a straight line, then the rate of change of y is not constant but varies with x. Therefore, the rate of change must itself be a function of x. This function is called the *derivative* of $\mathrm{f}(x)$. The process of finding the derivative is called *differentiation.*

In SAQ4-1-4 you were asked to measure the gradient of the curve $y = x^2$ at the points where $x = 1$, 0·5, and 2. If you had measured accurately (which is very difficult) you would have obtained the results 2, 1, and 4, respectively. This seems to imply that the gradient of the curve $y = x^2$ is equal to $2x$. That is in fact true, and the gradient of the curve at every point is $2x$.

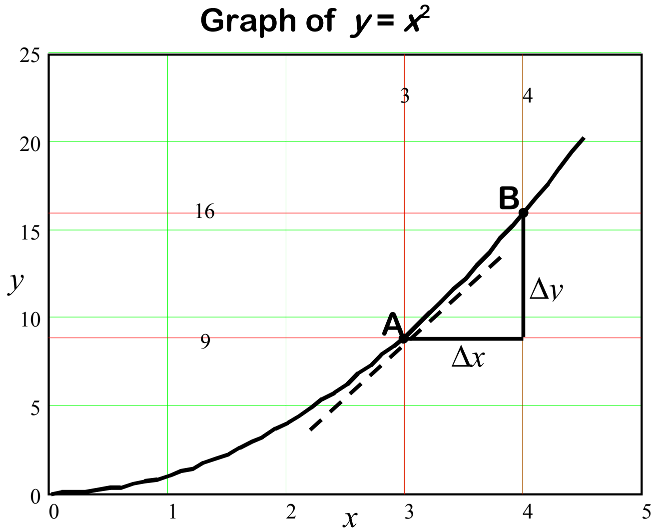
Therefore, the derivative of x^2 is equal to $2x$. We shall prove this on a subsequent page.

Differentiation from first principles

You have seen the difficulty of drawing tangents accurately and measuring their gradient. There is a similar difficulty in finding the gradient mathematically, and to do so we have to introduce the concept of a *limit*.

Consider the graph of $y = x^2$ as shown in the diagram below (not to scale).

Suppose we wish to find the rage of change (gradient) at the point A; ($x = 3$, $y = 9$). We know that the answer should be $2 \times 3 = 6$.



The straight line, AB which cuts the curve, (called a chord), has a gradient **:**

$$\Delta y/\Delta x = \frac{16-9}{4-3} = 7$$

If we move the point B nearer to point A, the gradient of the chord becomes nearer to the gradient of the tangent. So, let us keep halving Δx and see what happens.

Δ*x*	**Δ*y***	**Δ*y* / Δ*x***
1	$4^2 - 3^2$	7
0·5	$3{\cdot}5^2 - 9$ = 3·23.5	6·5
0·25	$3{\cdot}25^2 - 9$ = 1·5625	6·25
0·125	$3{\cdot}125^2 - 9$ = 0·765625	6·125
0·0625	$3{\cdot}0625^2 - 9$ = 0·37890625	6·062

$\Delta y/\Delta x$ seems to be getting closer to 6.

Now make Δx very small, say –0·0001

0·0001	$3{\cdot}0001^2 - 9$ = 0·00060001	6·0001

$\Delta y/\Delta x$ is even closer to 6. As Δx approaches zero, $\Delta y/\Delta x$ appears to be approaching 6. The problem is; how do we find the exact value of $\Delta y/\Delta x$ at the point where $x = 3$? If we let Δx equal zero the chord AB becomes the tangent at A, but Δy and Δx both become zero and we cannot evaluate $0 \div 0$. As stated in section 1 Algebra, division by zero is not defined in the arithmetic of real numbers (nor complex numbers). As Δy and Δx become infinitesimally small, the ratio $\Delta y/\Delta x$ appears to be approaching a *limit*, in this case 6, although if we let $\Delta y = 0$, $\Delta x = 0$, the

Limits

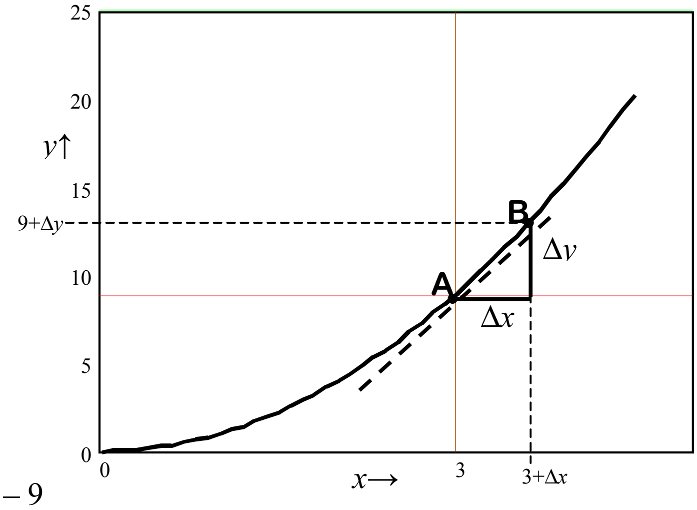
ratio cannot be evaluated. Expressions of the form 0 ÷ 0 are called *indeterminate* since they cannot be evaluated. However we can find the ***limit*** of $\Delta y / \Delta x$ as Δx approaches zero.

This is written as $$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

and is read as "the limit as Δx approaches zero, of $\Delta y / \Delta x$"

$\frac{dy}{dx}$ In calculus this limit is called $\frac{dy}{dx}$

How do we show, in the above example that $\frac{dy}{dx} = 6$ at $x = 3$?



At $x = 3$ we take an increment Δx.

$$\text{Then } \Delta y = (3 + \Delta x)^2 - 9$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{(3 + \Delta x)^2 - 9}{\Delta x}$$

$$= \frac{9 + 6\Delta x + (\Delta x)^2 - 9}{\Delta x}$$

$$= \frac{6\Delta x + (\Delta x)^2}{\Delta x}$$

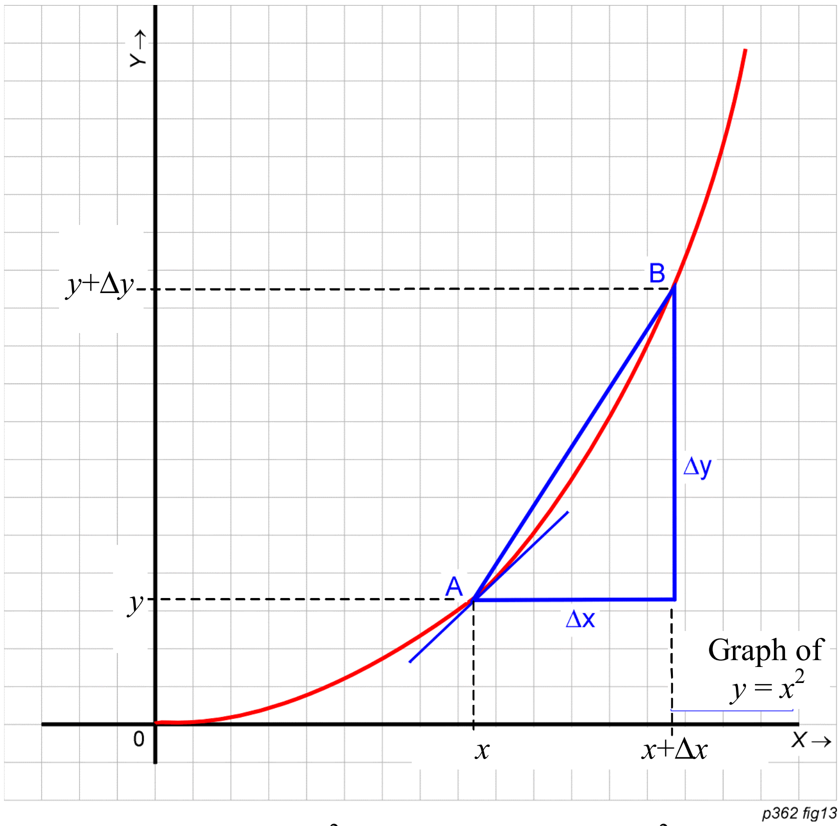
$$= 6 + \Delta x$$

Now if we let Δx approach zero we can see that in the limit, $\Delta y / \Delta x$ becomes equal to 6. Hence $\frac{dy}{dx} = 6$ at $x = 3$

This is an example of finding a **limit**. Other kinds of limits will be considered later in your course.

We can find a general formula for $\frac{dy}{dx}$ in terms of x by a similar method.

Consider some general point A, coordinates (x,y) on the curve $y = x^2$.
Take an increment Δx in x, giving a corresponding increment Δy in y to point B on the curve whose coordinates are $(x + \Delta x, y + \Delta y)$



Since point B is on the curve $y = x^2$, then $(y + \Delta y) = (x + \Delta x)^2$.

$$\therefore \quad \frac{\Delta y}{\Delta x} = \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

$$= \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x}$$

$$= \frac{2x\Delta x + (\Delta x)^2}{\Delta x}$$

$$= 2x + \Delta x$$

Now as Δx approaches zero, the chord AB approaches the tangent at A and we can see that in the limit approaches $2x$.

Hence if $y = x^2$, $\dfrac{dy}{dx} = 2x$.

The function $2x$ is called the *derivative* of the function x^2.

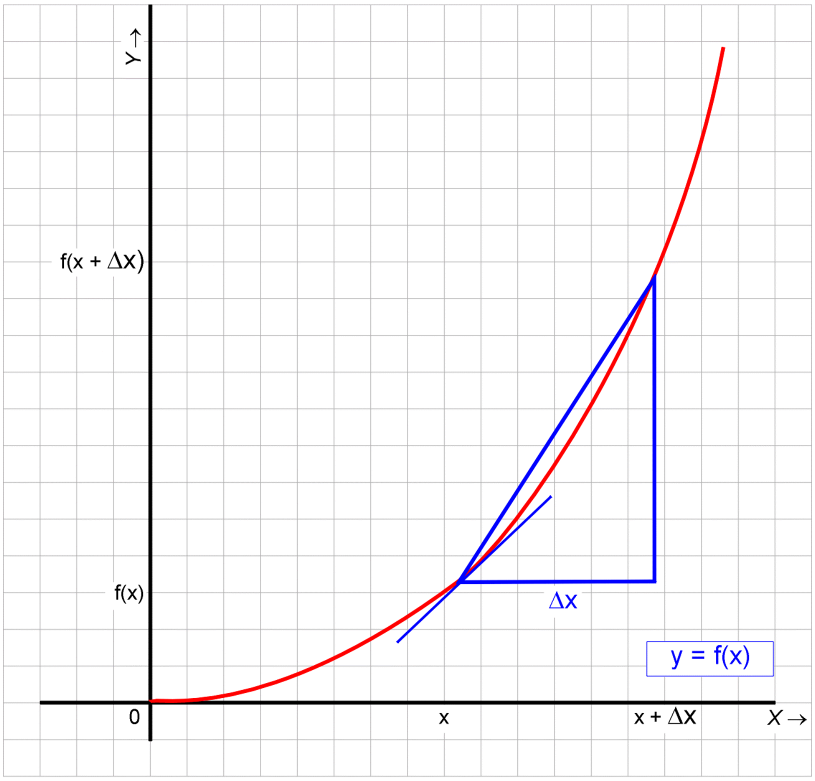
Other terms used for derivative are *differential coefficient* and *derived function*.

Function notation

If y = f(x) then $\frac{dy}{dx}$ is written as f ′(x).

Definition of derivative

We can now write a definition for the derivative of f(x) in terms of limits:



p362 fig14

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Instead of writing $^{dy}/_{dx}$ we can also write $^{d}/_{dx}$ f(x), treating $^{d}/_{dx}$ as an operator acting upon the function.

eg $\quad {}^{d}/_{dx}(x^2) = 2x$

The value of $^{dy}/_{dx}$ at $x = a$ is written f ′(a).

For example, if f(x) = x^2 then f ′(3) = 6.

The above method of finding the derivative by taking limits is known as "differentiating by first principles". Later we shall find short–cut methods for differentiating most functions.

SAQ3-2-1

a. If $f(x) = x^3$, using the same method as above, find $f'(x)$

{ Note the binomial expansion: $(a + b)^3 \equiv a^3 + 3a^2b + 3ab^2 + b^3$ }

b. Write down the values of:

(i) $f(2)$ (ii) $f'(2)$

Derivative of x^n

To save having to repeat a similar process every time we have to differentiate a function such as x^3, x^4, etc we can derive a general formula for the derivative of x^n, where n is a constant.

This proof is included for interest only and uses the binomial theorem which will not be taught until later on your course.

Let $f(x) = x^n$

Then from our definition of the derivative

$$f'(x) = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

Now, $(x + \Delta x)^n$ can be written as $x^n(1 + {}^{\Delta x}/_x)^n$

By the binomial theorem, since Δx is small, then for any value of n:

$$x^n(1+{}^{\Delta x}/_x)^n = x^n \quad \{1+n({}^{\Delta x}/_x)+\frac{n(m-1)}{2}({}^{\Delta x}/_x)^2+\frac{n(m-1)n-2)}{6}({}^{\Delta x}/_x)^3 + \bullet\bullet$$
$$\ldots + \text{higher powers of } \Delta x\}$$

$$= x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{6}x^{n-3}(\Delta x)^3 + \bullet\bullet$$
$$\ldots + \text{higher powers of } \Delta x$$

Subtracting x^n and dividing by Δx we get:

$$\lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left\{nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}\Delta x + \frac{n(n-1)(n-2)}{6}x^{n-3}(\Delta x)^2 + \ldots\right.$$
$$\ldots + \text{terms containing } \Delta x \}$$

We can see that as Δx approaches zero, all the terms containing Δx disappear so that it approaches the limit of nx^{n-1}.

Hence $f'(x) = nx^{n-1}$

This is true for any constant n; positive, negative, integer, or fraction.

The derivative of x^n is very important since many common functions such as polynomials contain expressions of this kind. This result should be committed to memory. It is restated below.

$$\text{if } y = x^n \quad \text{then} \quad \frac{dy}{dx} = nx^{n-1}$$

Examples

a. $y = x^3$, $\quad {}^{dy}/_{dx} = 3\,(x^{3-1}) \quad = 3x^2$

b. $y = x^4$, $\quad {}^{dy}/_{dx} = 4\,(x^{4-1}) \quad = 4x^3$

c. $y = x$, $\quad {}^{dy}/_{dx} = 1(x^{1-1}) \quad = 1(x^0) \quad = 1$

d. $y = x^{-1}$, $\quad {}^{dy}/_{dx} = -1\,(x^{-1-1}) \quad = -x^{-2}$

e. $y = x^{½}$, $\quad {}^{dy}/_{dx} = ½\,(x^{½-1}) \quad = ½x^{-½}$

Note that n can be any **constant;** positive, negative or fractional.

When we write ${}^{dy}/_{dx}$ we are differentiating *with respect to x,* i.e. x is the independent variable and we are finding the **rate of change with** y **with respect to** x.

Derivative of a constant

The graph of $y = C$ where C is a constant, is a line parallel to the x axis which has zero gradient. Therefore, if y is constant, ${}^{dy}/_{dx} = 0$.

This is consistent with the above rule, since we can regard a constant C as being Cx^0, since $x^0 = 1$.

Therefore ${}^{d}/_{dx}\,(x^0) = 0(x^{0-1}) = 0$

SAQ3-2-2

Write down $^{dy}/_{dx}$ for the functions of x in the table.

	y	$^{dy}/_{dx}$
a.	x^5	
b.	x^{-2}	
c.	$x^{-½}$	
d.	$x^{3/2}$	
e.	4	

Differentiation as a linear operation

Differentiation is a linear operation.

This means that the derivative of the sum of 2 functions is equal to the sum of the derivatives. ie

If $f(x) = f_1(x) + f_2(x)$

then $f'(x) = f_1'(x) + f_2'(x)$

ALSO if k is a constant then $^{d}/_{dx}\, k\, f(x) = k\, ^{dy}/_{dx}\, f(x)$

This means that when we have several functions added together, all we have to do is differentiate them separately. Also, a multiplicative constant may be taken outside the derivative.

Examples

If $y = x^3 + x^2$ then $^{dy}/_{dx} = 3x^2 + 2x$

i.e. simply differentiate term by term.

If $y = 5x^2$ then $^{dy}/_{dx} = 5(2x) = 10x$

i.e. the constant simply multiplies the derivative.

Examples

$y = 2x^3 - 4x^2 + 3x + 7,\ ^{dy}/_{dx} = 6x^2 - 8x + 3$

$y = 2x^{½} + 2x^{-1},\ ^{dy}/_{dx} = x^{-½} - 2x^{-2}$

SAQ3-2-3

Differentiate the following functions with respect to x.
(If necessary, refer to the table of derivatives on page 2–18)

a. $y = 2x^2 - x + 2$

b. $y = 4x^5 + 2x^3 - 5x^2 - 3x - 7$

c. $y = 1/x^3$

d. $y = \sqrt{x}$

e. $y = 2\sqrt{x} - \sqrt{(x^3)}$

f. $y = (x - 3)^2$

g. $y = x^2 - 1/x^2$

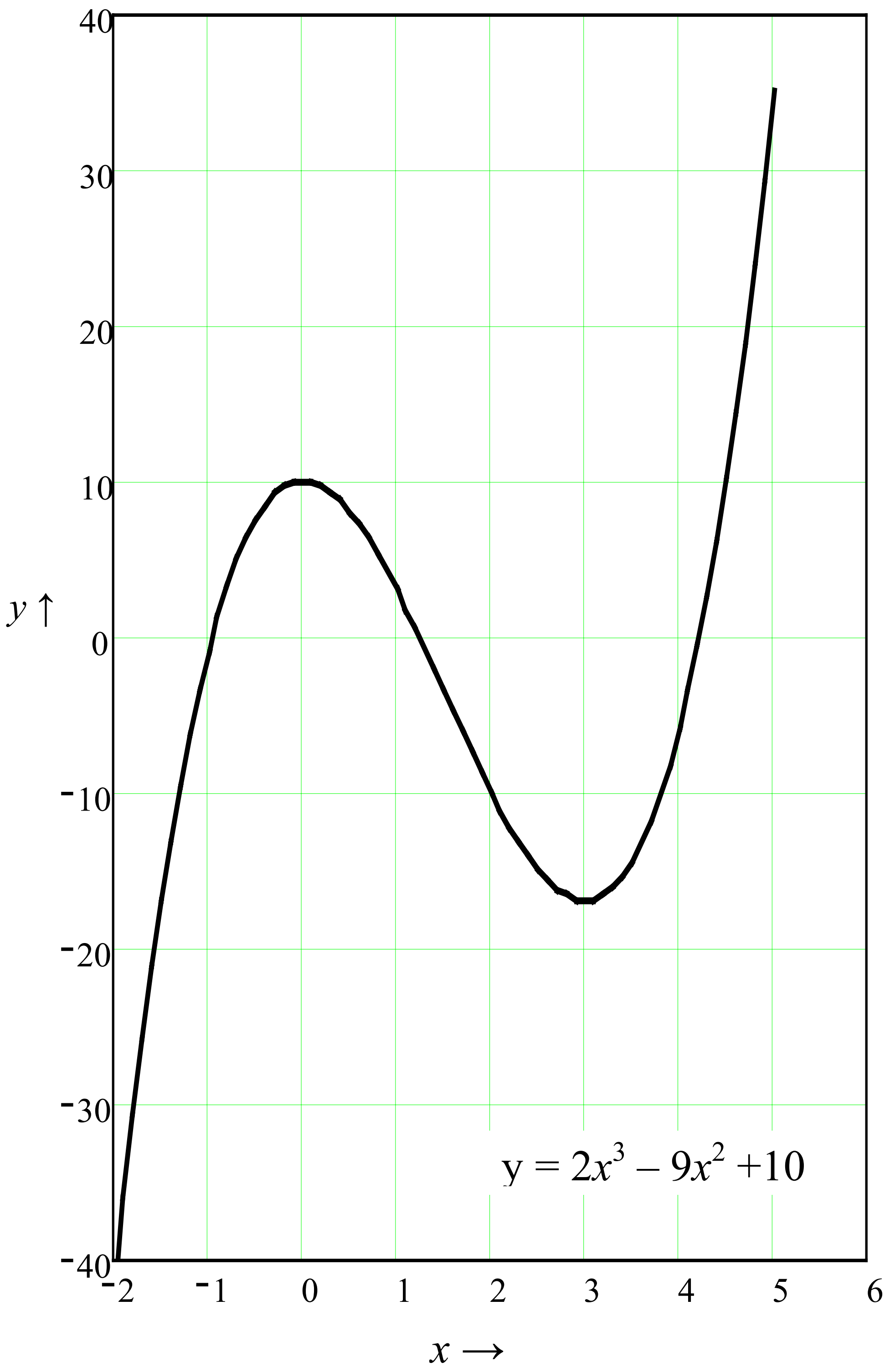
h. $y = 1\text{n}(x^2)$

SAQ3-2-4

Find the gradient of the curve $y = 2x^3 - 9x^2 + 10$ at the points

a. (2, –10) b. (0, 10) c. (3, –17) d. (–1, –1)

and mark these points on the graph below.



Further examples of limits

Try the following exercise.
Using a scientific calculator, select the "radian" mode for angles. Enter a small angle and calculate its sine. Then divide sin *x* by *x*, as shown in the table below.

Angle *x* (radians)	sin *x*	(sin *x*)/*x*
0·5	0·4794	0·9589
0·1	0·09983	0·9983
0·01	0·00999983	0·999983
0·001	0·00099999983	0·99999983
0·0001	0·000099999999	0·99999999
0·00001	0·000100000	1·000000000

We can see that as the angle gets smaller, the value of $\frac{\sin x}{x}$ gets closer to 1.

Eventually, the calculator runs out of available digits and it shows the value as 1, to the limit of its accuracy.

However if we put $x = 0$ we obtain $\frac{\sin x}{x} = 0 \div 0$, which cannot be evaluated.

It can be shown that $\frac{\sin x}{x}$ approaches the value 1, as *x* approaches zero.

This is a very important limit which should be remembered. You will encounter it later in signal processing and in antenna theory. It is written as:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

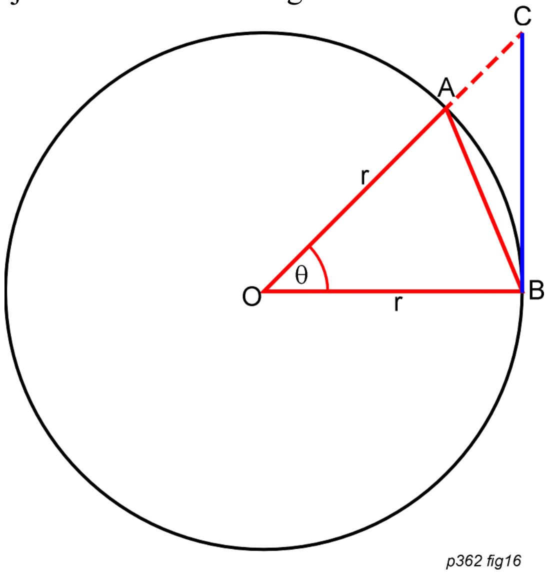
The angle *x* is, of course, in radians.

A proof of this limit is given on the next page. This proof is given for interest only and need not be memorised. The result, however, is very important.

Limits will be discussed further on your course at the Royal School of Signals.

Proof that $\lim_{x \to 0} \frac{\sin x}{x} = 1$

OAB is a sector of a circle with an angle θ (radians) at the centre.
BC is a tangent to the circle at B. A tangent to a circle makes a right angle with the radius, in this case with the radius OB.
The radius OA is projected to meet the tangent at C.



Let the radius of the circle = r.

From elementary trigonometry:
The area of the triangle OAB = $\frac{1}{2}r^2 \sin\theta$
The area of the sector OAB = $\frac{1}{2}r^2\theta$ (θ measured in *radians*)
The area of the triangle OCB = $\frac{1}{2}r^2 \tan\theta$

Hence, it is clear that $\frac{1}{2}r^2 \sin\theta \le \frac{1}{2}r^2\theta \le \frac{1}{2}r^2 \tan\theta$

∴ $\sin\theta \le \theta \le \tan\theta$ (divided by $\frac{1}{2}r^2$ which must be positive).

$$1 \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta}$$ (divided by sin θ, which is positive for small θ)

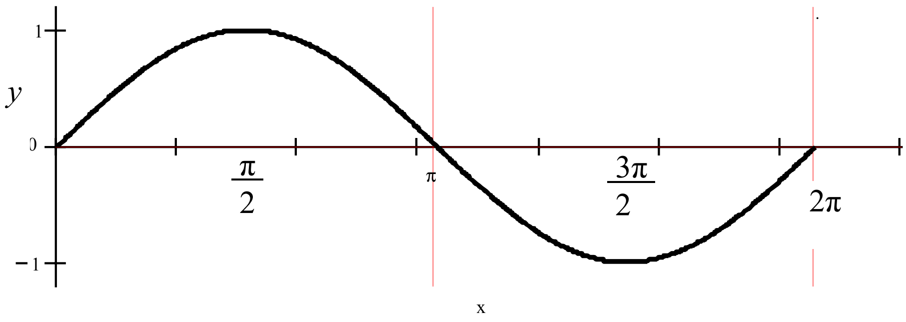
Now let θ approach zero so that the 3 areas converge together and cos θ approaches cos(0) = 1.
As θ→0:

$$1 \le \frac{\theta}{\sin\theta} \le 1$$

∴ $\frac{\theta}{\sin\theta}$ must approach the value 1.

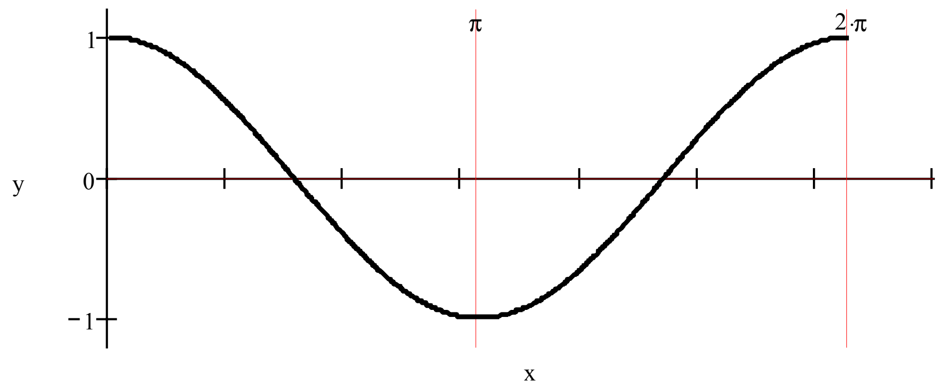
Derivative of trigonometric functions

We shall use the above limit to find the derivative of sin x.
Firstly, let us examine the graph of $y = \sin x$.



If we measure accurately the gradient of this curve at various points we get the following results. Note that the x axis is plotted in radians, not degrees.

x	**gradient ($^{dy}/_{dx}$)**
0	1
π/4	0·707
π/2	0
3π/4	–0·707
π	–1
5π/4	–0·707
3π/2	0
7π/4	0·707
2π	1



If we plot the graph of this gradient we obtain what must be a periodic function. It looks remarkably like a cosine curve. This is no coincidence since the derivative of sin x is, in fact, cos x.
A proof of this is given below. Later in this section there will be another proof by a different method.

Derivative of sin x

The following proof is for interest only and need not be learned.

Let $f(x) = \sin x$

Then by definition $f'(x) = \lim_{\Delta x \to 0} \dfrac{\sin(x+\Delta x) - \sin x}{\Delta x}$

Applying the trigonometric identity:

$\sin A - \sin B \equiv 2 \cos \tfrac{1}{2}(A + B) \ \sin \tfrac{1}{2}(A - B)$

we obtain

$\sin(x + \Delta x) - \sin x \equiv 2 \cos(x + {}^{\Delta x}/_2) \ \sin({}^{\Delta x}/_2)$

$$\therefore \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2\cos\left(x + \frac{\Delta x}{2}\right)\sin\left(\frac{\Delta x}{2}\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \cos\left(x + \frac{\Delta x}{2}\right) \frac{\sin\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}}$$

Now from the previously proved limit: $\lim_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$, where θ is in radians.

$\therefore \dfrac{\sin\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}}$ approaches 1 as Δx approaches zero.

Also, the ${}^{\Delta x}/_2$ in the bracket disappears and so

$f'(x) = \cos x$

Hence, ${}^{d}/_{dx} \sin x = \cos x$

Note that x is always in *radians*.

Derivative of cos x

By a similar method it can be proved that

${}^{d}/_{dx} \cos x = -\sin x$

Derivatives of other functions

An important derivative is that of e^x.
e^x is the function whose rate of change is equal to the value of the function at any instant, ie

Derivative of e^x

$$\frac{d}{dx} e^x = e^x$$

Derivative of $\ln x$

Another important derivative is that of the natural logarithm, $\ln x$.

$$\frac{d}{dx} \ln x = 1/x$$

Table of derivatives

A table of some common derivatives is given below.

y	$\frac{dy}{dx}$
x^n	nx^{n-1}
e^x	e^x
$\ln x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\mathrm{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\mathrm{cosec}\, x$	$-\mathrm{cosec}\, x \cot x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\mathrm{sech}^2 x$

Chapter 3

Differentiation; product and quotient rule

Diferentiation of a product

We have seen that the derivative of a sum is equal to the sum of the derivatives. However, this does not work for products, ie the derivative of a product is **not** the product of the derivatives.

The product rule is as follows:

$$\text{If } y = uv$$
$$\text{where } u, v \text{ are functions of } x$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

Examples

1. $y = x^2 \sin x$

   Let $u = x^2$ then ${}^{du}/_{dx} = 2x$

   Let $v = \sin x$ then ${}^{dv}/_{dx} = \cos x$

   ${}^{dy}/_{dx} = x^2 \cos x + 2x \sin x$

2. $y = x\,\mathrm{e}^x$

   Let $u = x$ then ${}^{du}/_{dx} = 1$

   Let $v = \mathrm{e}^x$ then ${}^{dv}/_{dx} = \mathrm{e}^x$

   ${}^{dy}/_{dx} = x\,\mathrm{e}^x + \mathrm{e}^x$

With a bit of practice, you should be able to write down the answers directly without the intermediate steps.

3. $y = \mathrm{e}^x \cos x$

   Let $u = x^2$ then ${}^{du}/_{dx} = 2x$

   Let $v = \sin x$ then ${}^{dv}/_{dx} = \cos x$

   ${}^{dy}/_{dx} = \mathrm{e}^x \cos x - \mathrm{e}^x \sin x$

If a product contains more than 2 factors, they must be grouped in pairs and the product rule applied more than once.

Example

$y = 2x^3 \, e^x \cos x$

Group 2 of the factors together

Let $u = 2x^3$, $\quad v = (e^x \cos x)$

${}^{du}/_{dx} = 6x^2$, $\quad {}^{dv}/_{dx} = e^x \cos x - e^x \sin x$

$\therefore \; {}^{dy}/_{dx} = 2x^3 \, e^x(\cos x - \sin x) + 6x^2 \, e^x \cos x$

SAQ3-3-1

Find ${}^{dy}/_{dx}$ where:

a. $y = 3x^2 \tan x$

b. $y = x \ln x - x$

c. $y = e^x \sin x \cos x$

Differentiation of a quotient

The quotient rule is as follows:

$$\text{If } y = \frac{u}{v}$$

where *u, v* are functions of *x*

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

In this formula, unlike the product formula, it is essential to have the *u* and *v* the correct way round.

Example

$$y = \frac{2x^3 - x}{x^2 + 1}$$

Let $u = 2x^3 - x$ then $^{du}/_{dx} = 6x^2 - 1$

Let $v = x^2 + 1$ then $^{dv}/_{dx} = 2x$

$$^{dy}/_{dx} = \frac{(x^2+1)(6x^2-1) \quad - \quad (2x^3-x)2x}{(x^2+1)^2}$$

$$= \frac{2x^4 + 7x^2 - 1}{(x^2+1)^2}$$

Example

Differentiate with respect to x

$$\frac{x \sin x}{x+2}$$

This contains both a product and a quotient.

Let $u = x \sin x$ then $^{du}/_{dx} = x \cos x + \sin x$, by the product rule.

Let $v = x + 2$ then $^{dv}/_{dx} = 1$

$$^{dy}/_{dx} = \frac{(x+2)(x\cos x + \sin x) - x \sin x}{(x+2)^2}$$

$$= \frac{(x+2)x\cos x + 2\sin x}{(x+2)^2}$$

SAQ3-3-2

Differentiate with respect to x, the following functions.

a. $\dfrac{x^3 - x^2 + 3}{2x+1}$

b. $\dfrac{x^2}{3x+5}$

c. $\dfrac{5x^2 e^x}{1+x^2}$

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Chapter 4

Differentiation; function of a function

Chain rule	So far we have only considered simple functions of x such as polynomials and single trigonometric functions.

A function of a function is an expression of the type $F\{f(x)\}$ where $f(x)$ is a function of x and $F\{f(x)\}$ is a function of $f(x)$.
For example.

$$y = \sqrt{x^2 + 1}$$

$x^2 + 1$ is a function of x and $\sqrt{x^2 + 1}$ is a function of $x^2 + 1$.

$$y = e^{2x}$$

$2x$ is a function of x and e^{2x} is a function of $2x$.

These functions cannot be differentiated by any of the rules we have used so far. To differentiate a function of a function we use the chain rule which is:

If y is a function of z where z is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{dz}\ \frac{dz}{dx}$$

This rule is very easy to remember since it appears that we are "cancelling" the dz. This is not quite true, since a derivative is not a ratio but the limit of a ratio. A rigorous proof of the above rule is beyond the scope of this course.

Example

$$y = \sqrt{x^2 + 1}$$

Let $z = x^2 + 1$, $\quad y = z^{1/2}$

$dz/dx = 2x$, $\quad dy/dz = \frac{1}{2} z^{-1/2} = \frac{1}{2}(x^2 + 1)^{-1/2}$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \tfrac{1}{2}(x^2 + 1)^{-1/2}(2x)$$

$$= x(x^2 + 1)^{-1/2}$$

$$= \frac{x}{\sqrt{x^2 + 1}}$$

Example	$y = \sin^2 x$
	Let $z = \sin x$, $\quad y = z^2$
	${}^{dz}/_{dx} = \cos x$, $\quad {}^{dy}/_{dz} = 2z = 2\sin x$
	${}^{dy}/_{dx} = {}^{dy}/_{dz} \quad = \quad 2\sin x \cos x$
Example	$y = \ln(x^2 + 1)$
	Let $z = x^2 + 1$, $\quad y = \ln z$
	${}^{dz}/_{dx} = 2x$, $\quad {}^{dy}/_{dz} = 1/z = 1/(x^2 + 1)$
	${}^{dy}/_{dx} = {}^{dy}/_{dz}\ {}^{dz}/_{dx} = \dfrac{2x}{x^2+1}$
Derivative of e^{ax}	A very important derivative is that of e^{ax} where a is a constant.
	$y = e^{ax}$
	Let $z = ax$, $\quad y = e^z$
	${}^{dz}/_{dx} = a$, $\quad {}^{dy}/_{dz} = e^z = e^{ax}$
	${}^{dy}/_{dx} = {}^{dy}/_{dz}\ {}^{dz}/_{dx} \quad = ae^{ax}$
	e.g. ${}^{d}/_{dx}\,(e^{2x}) = 2e^{2x}$
Deriviative of $\sin \omega t$	Another important derivative is that of $\sin ax$ or $\sin \omega t$
	$y = \sin \omega t$
	Let $z = \omega t$, $\quad y = \sin z$
	${}^{dz}/_{dt} = \omega$, $\quad {}^{dy}/_{dz} = \cos z = \cos \omega t$
	${}^{dy}/_{dt} = {}^{dy}/_{dz}\ {}^{dz}/_{dt} \quad = \omega \cos \omega t$
	Which shows that the rate of change of a sine wave is directly proportional to its frequency.

Example

The derivative of the sine or exponential of any linear function of x is similar.

$$y = e^{ax+b}$$

Let $z = ax+b$, $\quad y = e^z$

$dz/dx = a \quad dy/dz = e^z = e^{ax+b}$

$dy/dx = dy/dz \; dz/dx \quad = ae^{ax+b}$

Example

$$y = \sin(\omega t + \phi)$$

Let $z = \omega t + \phi \quad y = \sin z$

$dz/dx = \omega \quad dy/dz = \cos z = \cos(\omega t + \phi)$

$dy/dx = dy/dz \; dz/dx \quad = \omega \cos(\omega t + \phi)$

After some practice you should be able to write down the answers without having to go through the intermediate substitutions.

SAQ3-4-1

Find the derivatives, with respect to x, of the following functions:

a. $\sqrt{2x^2 + 4x}$

b. $\tan^2 x$

c. $\ln(x^3 + 3x)$

d. e^{2x+1}

e. $\ln(\sec x + \tan x)$

f. $(3x + 2)^{12}$

g. $\sin(2x + \pi/6)$

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Extension of chain rule

The chain rule can be extended to more complicated functions of functions of functions, ie

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}$$

Example

$y = \sqrt{(\sin 2x)}$

$v = 2x,$ $\quad u = \sin v,$ $\quad y = u^{½}$

${}^{dv}/_{dx} = 2,$ $\quad {}^{du}/_{dv} = \cos v = \cos 2x,$ $\quad {}^{dy}/_{du} = ½u^{-½} = ½(\sin v)^{-½} = ½(\sin 2x)^{-½}$

$${}^{dy}/_{dx} = {}^{dy}/_{du}\ {}^{du}/_{dv}\ {}^{dv}/_{dx}$$

$$= 2 \times \cos 2x \times ½(\sin 2x)^{-½}$$

$$= \frac{\cos 2x}{\sqrt{(\sin 2x)}}$$

SAQ3-4-2

Find ${}^{dy}/_{dx}$ where $y = \ln(\cos 4x)$

Sometimes a problem has to be split into separate parts

Example

$$y = \ln(x + \sqrt{x^2 + 1})$$

Let $z = x + \sqrt{x^2 + 1}$ $\qquad y = \ln z$

$$dy/dz = 1/z = 1/(x + \sqrt{x^2 + 1})$$

Now, $\sqrt{x^2 + 1}$ is itself a function of a function

Let $u = \sqrt{x^2 + 1}$, $\qquad v = x^2 + 1$, $\qquad u = v^{1/2}$

$$du/dx = du/dv \; dv/dx = \tfrac{1}{2}v^{-1/2}(2x) = x(x^2 + 1)^{-1/2}$$

Hence, $dz/dx = 1 + x(x^2 + 1)^{-1/2}$

$$= 1 + \frac{x}{\sqrt{x^2 + 1}}$$

$$dy/dx = dy/dz \; dz/dx$$

$$= \frac{1 + \frac{x}{\sqrt{x^2 + 1}}}{x + \sqrt{x^2 + 1}}$$

This may be simplified by various methods, eg

$$\frac{1 + \frac{x}{\sqrt{x^2 + 1}}}{x + \sqrt{x^2 + 1}}$$

$$= \frac{\frac{1}{\sqrt{x^2 + 1}} \left\{\sqrt{x^2 + 1} + x\right\}}{x + \sqrt{x^2 + 1}}$$

$$= \frac{1}{\sqrt{x^2 + 1}}$$

Derivative of $\sin x$

We can use the function of a function rule to find the derivatives of sine and cosine.

You will recall from **Section 2: Complex numbers** that

$$\sin x = \frac{e^{jx} - e^{-jx}}{2\,j}$$

SAQ3-4-3

$$\cos x = \tfrac{1}{2}(e^{jx} + e^{-jx})$$

By differentiating this expression, show that $\frac{d}{dx}(\cos x) = -\sin x$

Chapter 5

Higher derivatives

Second derivative

The derivative of y with respect to x is the *rate of change* of y with respect to x. In mechanics, ${}^{ds}/_{dt}$ is the rate of change of distance, s, with respect to time, t. This is called velocity, v. The rate of change of velocity with respect to time is called acceleration, a. Hence, $a = {}^{dv}/_{dt}$.

$$\therefore \quad a = {}^{d}/_{dt} \quad ({}^{ds}/_{dt})$$

This is the *second derivative* of s with respect to t and is written ${}^{d^2s}/_{dt^2}$.

Similarly, $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ is written as $\dfrac{d^2y}{dx^2}$.

It is said as "Dee two y, dee x squared" but note that it is not actually x squared and is **not** the derivative with respect to x^2.

${}^{dy}/_{dx}$ is sometimes called the *first derivative.*

Examples

$y = x^3, \quad {}^{dy}/_{dx} = 3x^2. \quad {}^{d^2y}/_{dx^2} = 6x$

$y = \sin \omega t, \quad {}^{dy}/_{dt} = \omega \cos \omega t, \quad {}^{d^2y}/_{dx^2} = -\omega^2 \sin \omega t$

Third derivative

The derivative of the second derivative is called the *third derivative*, and is written as ${}^{d^3y}/_{dx^3}$. Similarly the derivative of the third derivative is called the fourth derivative, etc.

In function notation the first derivative is written $f'(x)$. The higher derivatives are written in a similar manner:

Second derivative	$f''(x)$
Third derivative	$f'''(x)$
Fourth derivative	$f''''(x)$
Fifth derivative	$f^{v}(x)$
Sixth derivative	$f^{vi}(x)$
.	.
.	.
n^{th} derivative	$f^{(n)}(x)$

SAQ3-5-1

If $y = f(x) = x^4 + 4x^3 + 2x^2 - 2x + 1$, find

a. dy/dx

b. d^2y/dx^2

c. d^3y/dx^3

d. $f'(2)$

e. $f''(1)$

f. $f'''(-1)$

SAQ3-5-2

The distance s metres of a body, moving in a straight line, from a fixed point, at time t seconds, is given by

$$s = 5t^4 - 14t^3 + 8t^2$$

If velocity $= \mathrm{d}s/\mathrm{d}t$, acceleration $= \mathrm{d}^2 s/\mathrm{d}t^2$

Find:

a. The 2 times after $t = 0$ when the body is again passing through its point of origin, and its velocity and acceleration at these 2 instants.

b. The times at which the velocity is zero, and its acceleration at these instants.

Chapter 6

Integration

Integration

Integration has many important applications in electrical theory and signal processing. Originally, integration was derived as a method of finding areas but was then proved to have a relationship to differentiation. For most purposes, integration may be regarded as the reverse process to differentiation. For all of the elementary continuous functions which we shall encounter, integration can be performed in this way.

In the previous chapters, we had to find $^{dy}/_{dx}$ given y. Suppose we are given $^{dy}/_{dx}$ and asked to find y. For example:

$$^{dy}/_{dx} = 2x, \text{ find } y$$

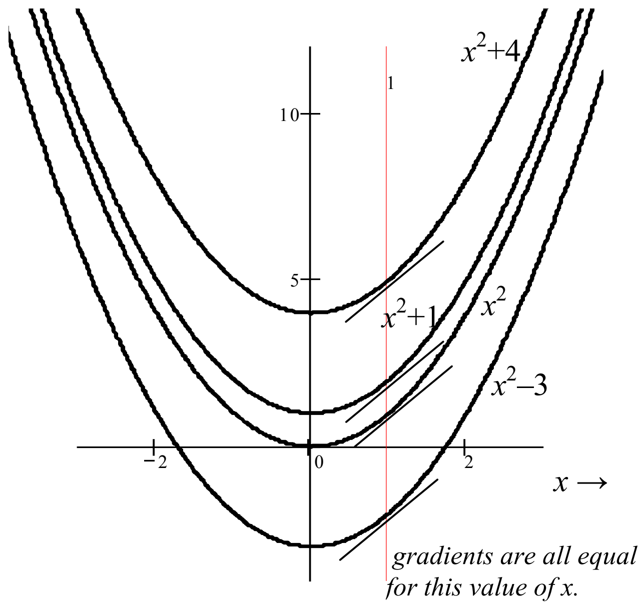
We know that if you differentiate $y = x^2$, you get $^{dy}/_{dx} = 2x$, so we could say that the answer is $y = x^2$. However if you differentiate $y = x^2 + 1$ you also obtain $^{dy}/_{dx} = 2x$. In fact if you differentiate $y = x^2 + \mathbf{C}$ where $\mathbf{C}$ is *any* constant, you obtain $^{dy}/_{dx} = 2x$.

$$\text{Therefore we write } y = x^2 + \mathbf{C}$$

This is illustrated in the graph below.

For some particular value of x, all these curves have the same gradient, since $^{dy}/_{dx}$ is equal to $2x$ for all of them.

Therefore, given $^{dy}/_{dx}$ we cannot determine y exactly.



Arbitrary constant

$\mathbf{C}$ is called an arbitrary constant because it can take any value. We cannot determine the value of this constant unless we are given additional information.

For example, suppose we are given the additional information that y=2 when x=1.

We have $y = x^2 + \mathbf{C}$

Substituting x=1, y=2 we get $2 = 1^2 + \mathbf{C}$

$\therefore \mathbf{C} = 1$

Hence $y = x^2 + 1$

Symbol for integration

The symbol for integration is an elongated S. Thus we write

$$\int 2x \, dx = x^2 + \mathbf{C}$$

The dx indicates that we are integrating with respect to x and must not be left out. This type of integral is called an indefinite integral because it contains an arbitrary constant. The arbitrary constant must not be omitted, since, as you will discover in applications to circuit theory, the arbitrary constant has a particular meaning.

To integrate simple functions we can simply use differentiation in reverse.

We know that $^{d}/_{dx}\,(x^n) = nx^{n-1}$ therefore if we integrate x^n the power must increase by 1. It is clear that

Integral of x^n

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + \mathbf{C}$$

We can check this by differentiating back again

$$\frac{d}{dx}\left\{\frac{x^{n+1}}{n+1}\right\} = \frac{(n+1)x^{n+1-1}}{n+1} = x^n$$

This integral is true for any value of n, positive, negative, or fractional,
except for n=–1.

If we put n=–1 we get $x^0 \div 0 = 1 \div 0$. This cannot be correct. We know that $^{d}/_{dx}\,(\ln x) = x^{-1}$, hence:

$$\int x^{-1} \, dx = \ln x + \mathbf{C}$$

Note that $\ln x + \mathbf{C}$ can also be written as $\ln(\mathbf{K}x)$

where by the rules of logarithms, $\mathbf{C} = \ln \mathbf{K}$ (cf Section 1: Algebra)

If **C** is an arbitrary constant then **K** must also be an arbitrary constant. As we do not know what the constant is, it does not matter what we call it (**A, B, C,** etc). In this text we shall use capital letters to denote arbitrary constants, avoiding letters such as **X, Y** which we commonly use for variables.

Table of standard integrals

Most of the common integrals can be found simply by looking at our standard derivatives. A table of standard integrals is given below.

$f(x)$	$\int f(x)\,dx$	
x^n	$\dfrac{x^{n+1}}{n+1} + \mathbf{C}$	$(n \neq -1)$
$\dfrac{1}{x}$	$\ln x + \mathbf{C}$	
e^x	$e^x + \mathbf{C}$	
$\sin x$	$-\cos x + \mathbf{C}$	
$\cos x$	$\sin x + \mathbf{C}$	
$\tan x$	$\ln(\sec x) + \mathbf{C}$	
$\sec x$	$\ln(\sec x + \tan x) + \mathbf{C}$	
$\cot x$	$\ln(\sin x) + \mathbf{C}$	
$\mathrm{cosec}\, x$	$\ln(\tan \tfrac{1}{2}x) + \mathbf{C}$	
$\sec^2 x$	$\tan x + \mathbf{C}$	
$\sinh x$	$\cosh x + \mathbf{C}$	
$\cosh x$	$\sinh x + \mathbf{C}$	

∗ C is an arbitrary constant

SAQ3-6-1

Integrate the functions in the table below:

f(x)	$\int$ f(x)dx
x^5	
$\sqrt{x}$	
$1/\sqrt{x}$	
x^{-2}	
x	
3	

Integration as a linear operation

Since differentiation is a linear operation, integration must be a linear operation also, ie

If k is a constant then $\int k\,\mathrm{f}(x)\,\mathrm{d}x \;=\; k\int \mathrm{f}(x)\,\mathrm{d}x$

and

$$\int \{\mathrm{f}_1(x) + \mathrm{f}_2(x)\}\mathrm{d}x \;=\; \int \mathrm{f}_1(x)\,\mathrm{d}x + \int \mathrm{f}_2(x)\,\mathrm{d}x$$

Example

$$\int 2\cos x\,\mathrm{d}x \;=\; 2\int \cos x\,\mathrm{d}x \;=\; 2\sin x + \mathbf{C}$$

Example

$$\int 12x^2\mathrm{d}x \;=\; 12\int x^2\,\mathrm{d}x \;=\; 12(x^3/3) + \mathbf{C} \;=\; 4x^3 + \mathbf{C}$$

Example

$$\int (x^3 + 6x^2 + 2x + 4)\,\mathrm{d}x \;=\; \tfrac{1}{4}x^4 + 2x^3 + x^2 + 4x + \mathbf{C}$$

Example

$$\int (\cos x - \sin x)\,\mathrm{d}x \;=\; \sin x + \cos x + \mathbf{C}$$

Example

$$\int -x^{-3}\,\mathrm{d}x \;=\; \frac{-(x^{-2})}{-2} + \mathbf{C} \;=\; \tfrac{1}{2}x^{-\frac{1}{2}} + \mathbf{C}$$

$$= \frac{1}{2x^2} + \mathbf{C}$$

SAQ3-6-2

Determine the following integrals:

a. $\int (3x^5 + 8x^3 - 15x^2 + x - 1)\, dx$

b. $\int \frac{2}{x}\, dx$

c. $\int \frac{dx}{2\sqrt{x}}$

d. $\int \frac{2}{x^3}\, dx$

e. $\int (3 \cos x + 2 \sin x)\, dx$

f. $\int (x + {}^{1}/_{\sqrt{x}})\, dx$

More complicated integrals

Although it is possible to differentiate the most complicated expressions by using product, quotient, and chain rules, integration is not quite so easy. There is no general product rule and no quotient rule. Integrating functions of functions is not always possible and there are various techniques and standard integrals which will be taught later on your course at the Royal School of Signals.

In this text, we shall only consider integrating expressions which are functions of *linear* functions of x.

Exponent of a linear function

$\int e^{ax+b}\ dx$ where a, b are constant.

We know that $\frac{d}{dx}\, e^{ax+b} = a e^{ax+b}$
Hence, we deduce by the reverse operation that

$$\int e^{ax+b}\ dx = \frac{1}{a}\, e^{ax+b} + C$$

since, if we differentiate back again, the $\frac{1}{a}$ cancels the a in the derivative of e^{ax+b}.
Check this yourself by differentiating $\frac{1}{a}\, e^{ax+b}$.

Note that this only works for **linear** functions of x. For example

$\int e^{ax^2}\ dx$ cannot be found at all, by this, or any other method.

Examples

$$\int e^{2x+1}\ dx = \tfrac{1}{2}\, e^{2x+1} + C$$

$$\int e^{-4x+2}\ dx = -\tfrac{1}{4}\, e^{-4x+2} + C$$

$$\int e^{x/2}\ dx = 2\, e^{x/2} + C$$

Sine and cosine of linear functions	$\int \sin(\omega t + \phi)\,dt$

We know that $\frac{d}{dt}\cos(\omega t + \phi) = -\omega \sin(\omega t + \phi)$

Therefore by the reverse operation we deduce that

$$\int \sin(\omega t + \phi)\,dt = -\frac{1}{\omega}\cos(\omega t + \phi) + \mathbf{C}$$

You should check this by differentiating back again.

Example

$$\int \sin(2t + \pi/6)\,dt = -\tfrac{1}{2}\cos(2t + \pi/6) + \mathbf{C}$$

Similarly since $\frac{d}{dt}\sin(\omega t + \phi) = \omega \cos(\omega t + \phi)$
we can deduce that

$$\int \cos(\omega t + \phi)\,dt = \frac{1}{\omega}\sin(\omega t + \phi) + \mathbf{C}$$

Check this by differentiating back again.

Example

$$\int \cos(0\cdot 1t - 1\cdot 5)\,dt = 10\sin(0\cdot 1t - 1\cdot 5) + \mathbf{C}$$

SAQ3-6-3

Determine the following integrals.

a. $\int e^{2x}\,dx$

b. $\int e^{3x-2}\,dx$

c. $\int e^{-x}\,dx$

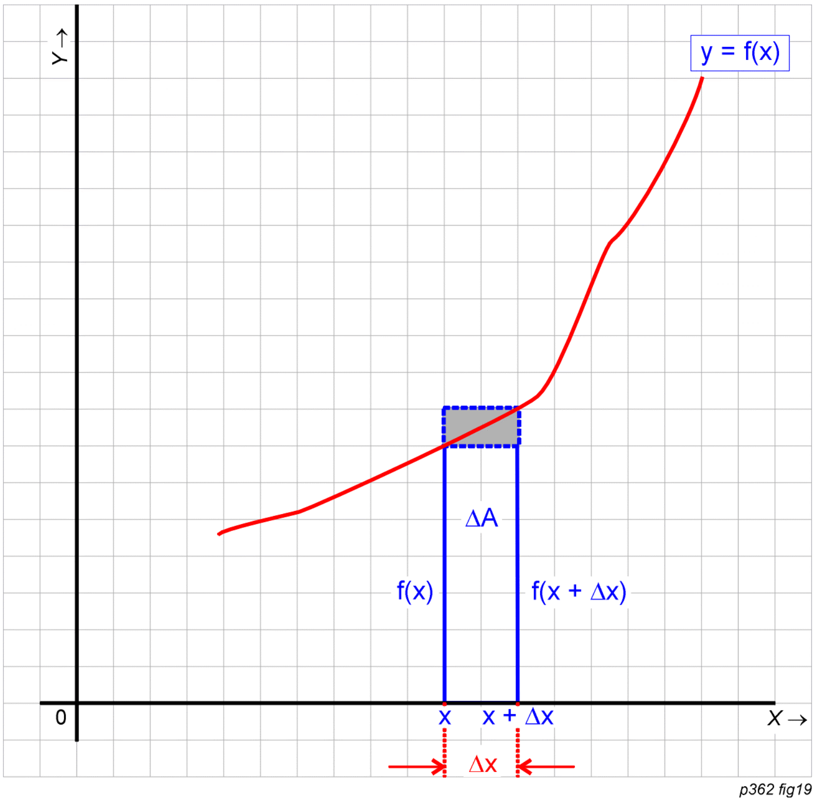
d. $\int \cos(2x + \pi/3)\,dx$

e. $\int \sin(0{\cdot}01t + 0{\cdot}5)\,dt$

Chapter 7

Definite integrals

Area under a curve	Consider the graph of $y = f(x)$ at some arbitrary point x.



Taking a small increment Δx in x, we obtain an increment in the area between the curve and the x axis which we shall call ΔA.

We can see that ΔA is greater than the area of the rectangle whose area is $f(x)\,\Delta x$ and that ΔA is less than the area of the rectangle whose area is $f(x+\Delta x)\,\Delta x$.

i.e. $$f(x)\,\Delta x \le \Delta A \le f(x+\Delta x)\,\Delta x$$

∴ $$f(x) \le \frac{\Delta A}{\Delta x} \le f(x+\Delta x)$$

Now suppose we let Δx approach zero. ${}^{\Delta A}/_{\Delta x}$ approaches ${}^{dA}/_{dx}$ and $f(x+\Delta x)$ approaches $f(x)$. Hence, in the limit

$$\frac{dA}{dx} = f(x)$$

ie the area is changing at a rate which at any point is equal to $f(x)$.

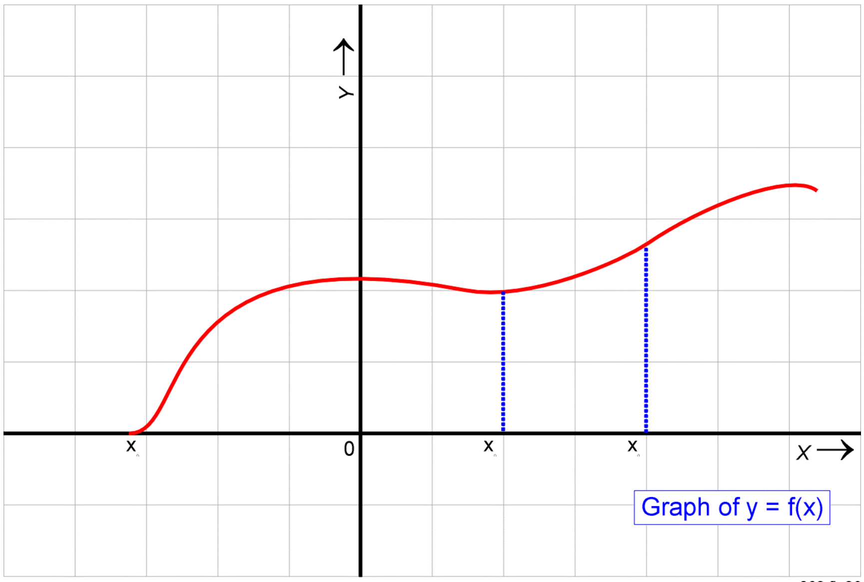
Therefore, the area A varies with x in accordance with some function

$$A = \int f(x)\,dx$$

Let us call this function $F(x) + \mathbf{C}$ which is the indefinite integral of $f(x)$ with respect to x.

The intregral contains the unknown constant **C** because the only information we have initially is the rate of change of area f(x) and we have an unspecified starting point from which to calculate the area.

We can assume that there is some unspecified point $x = x_0$ on the x axis, up to which the area is zero (this point may be $-\infty$). Suppose we wish to find the area between 2 values of x; $x = x_1$ and $x = x_2$.



p362 fig20

The area under the curve between $x = x_0$ and the ordinate at $x = x_1$ is equal to $F(x_1) + \mathbf{C}$.

The area under the curve between $x = x_0$ and the ordinate at $x = x_2$ is equal to $F(x_2) + \mathbf{C}$.

where

$$F(x) + \mathbf{C} = \int f(x)\,dx$$

Hence, the area bounded by the curve, the x axis and the ordinates at $x = x_1$ and $x = x_2$ is given by

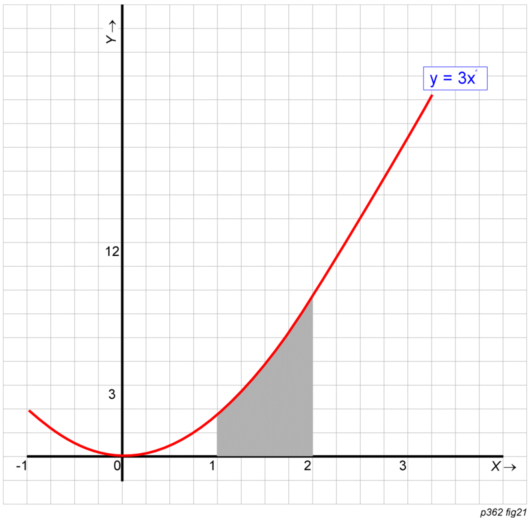
$$\{ F(x_2) + \mathbf{C} \} - \{ F(x_1) + \mathbf{C} \}$$

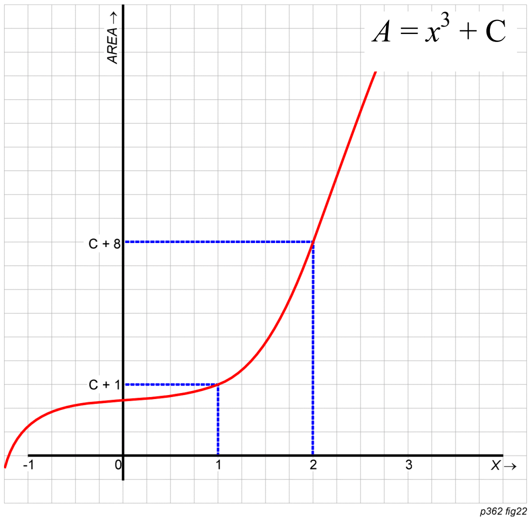
$$= F(x_2) - F(x_1)$$

This is written as

$$\int_{x_1}^{x_2} f(x)\,dx$$

Example





Consider the function $(x) = 3x^2$. We wish to find the area under the curve between the x axis and the 2 ordinates x=1 and x=2, as shown in the shaded area of the diagram.

We have shown that ${}^{dA}/_{dx} = f(x) = 3x^2$

$$\therefore \quad A = \int 3x^2\,dx = x^3 + \mathbf{C}$$

The graph of A against x is shown on the right. We do not know the value of **C** but this does not matter because the required area is the difference between 2^3 + **C** and 1^3 + **C** which is equal to 7.

We write this as $$\int_1^2 3x^2 dx$$

$$= \left[x^3\right]_1^2 = 2^3 - 1^3 = 7$$

The 2 numbers on the integral sign are called boundary values. The one at the top is called the upper limit of integration and the one at the bottom is called the lower limit of integration.

The square brackets mean "evaluate the function in brackets at the 2 limits, and subtract the value at the lower limit from the value at the upper limit."

Note that we don't usually write + **C** in the bracket because **C** has cancelled out, as it has the same unknown value at both points.

Definite integral	This type of integral is called a definite integral because the arbitrary constant disappears.
Examples	1. $\int_2^3 (8x^3 + x^2 - 4x + 2)\,dx$ $= \left[2x^4 + x^3/3 - 2x^2 + 2x\right]_2^3 = (162 + 9 - 18 + 6) - (32 + 8/3 - 8 + 4)$ $= 128\ ^1/_3$
	2. $\int_0^{\pi} \sin x\,dx = \left[-\cos x\right]_0^{\pi} = -\cos\pi - (-\cos 0)$ $= -(-1) - (-1) = 2$ *(Always remember: angles are radians in calculus)*
	3. $\int_0^{\infty} e^{-x}\,dx = \left[-e^{-x}\right]_0^{\infty} = 0 - (-1) = 1$
	4. $\int_1^2 \frac{dx}{x} = \left[\ln x\right]_1^2$ $= \ln 2 - \ln 1$ $= \ln 2 - 0$ $= \ln 2 \cong 0{\cdot}693$

SAQ3-7-1

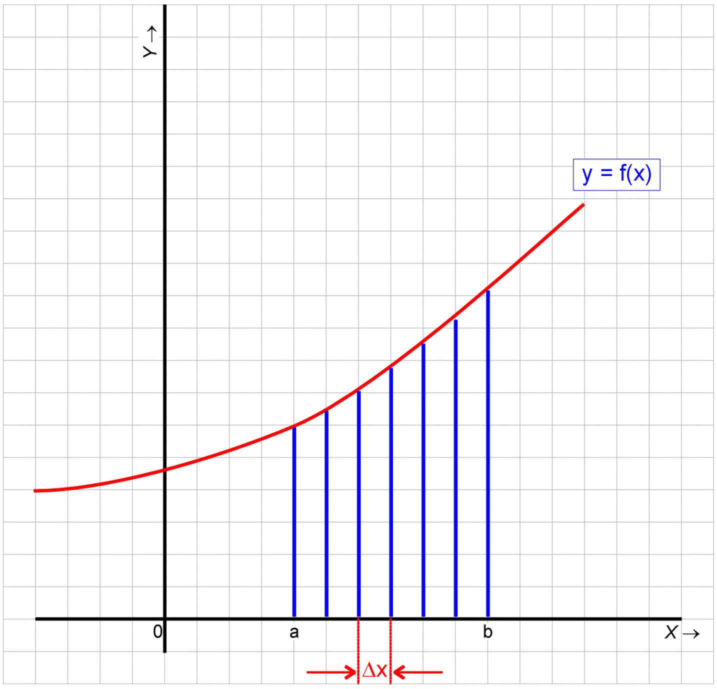
Evaluate the following definite integrals

a. $\int_1^3 (10x^3 - 2x^2 + 6x - 1)\,\mathrm{d}x$

b. $\int_0^{\pi/6} 2\cos x\,\mathrm{d}x$

c. $\int_1^4 3\sqrt{x}\,\mathrm{d}x$

Integral as a sum	Integration was originally derived as the sum of an infinite number of very small quantities.



If we divide the area between $x=a$ and $x=b$ into a number of narrow strips of width Δx, at some particular value of x, the strip is approximately a rectangle of height $f(x)$ and width Δx. Therefore the area ΔA of the strip is approximately given by

$$\Delta A \cong f(x)\,\Delta x$$

If we make the strip narrower, the error is assuming a rectangle becomes less. The total area is approximately given by

$$A \cong \sum_{x=a}^{x=b} f(x)\Delta x$$

The letter Σ is a short–hand way of writring "the sum of all such terms", ie

$f(x_1)\,\Delta x + f(x_2)\,\Delta x + f(x_3)\,\Delta x + f(x_4)\,\Delta x + \cdots$
from $x=a$ to $x=b$.

If we let Δx approach zero, so that we are summing an infinite number of infinitesimally small rectangles, then this sum approaches the area under the curve which is the integral of $f(x)$ from $x=a$ to $x=b$. Therefore

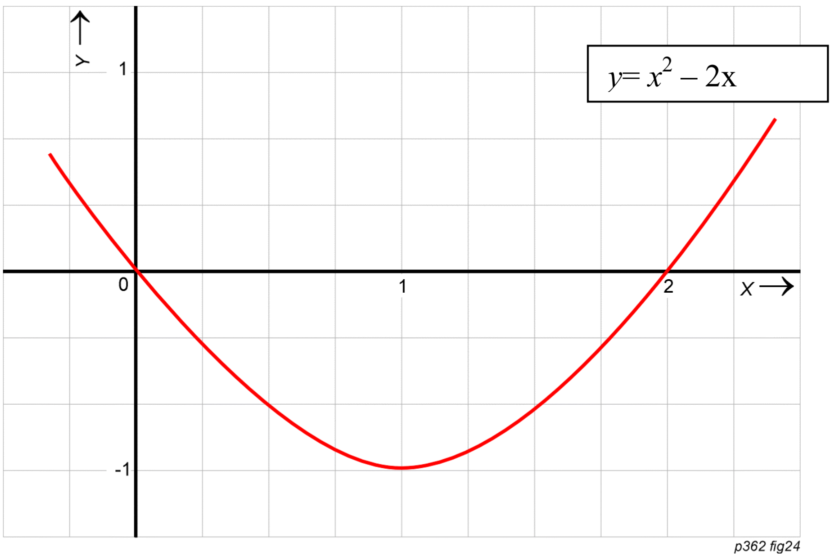
$$\lim_{\Delta x \to 0} \sum_{x=a}^{x=b} f(x)\Delta x = \int_a^b f(x)\,dx$$

The symbol used for integration which is a Gothic letter *S*, replaces the Greek letter *S* or Σ which stands for "sum".

Negative area

When we integrate, we are summing up the products of Δx, which is a positive increment in x, with values of y. If the values of y are negative then the product is negative. Therefore any area which lies below the x axis has a negative sign.

Find the area between the curve $y = x^2 - 2x$ and the x axis, between x=0 and x=2.



$$\int_0^2 (x^2 - 2x)\,dx \quad = \quad [x^3/3 - x^2]_0^2 \quad = \quad (7/3 - 4) \quad -0$$

$$= \quad = -1\,^1/_3$$

The negative sign merely indicates that the area lies below the x axis.

Units of an integral

Note that it is not necessary to write "square units" after the answer to a definite integral. The units of an integral are the product of the units on the x and y axes and unless the axes are labelled with units we cannot give specific units to the integral. However, in practical problems when units are given, it is essential to state the units of an integral (or of a derivative).

Example

y axis: velocity (m/s); x axis: time (seconds)

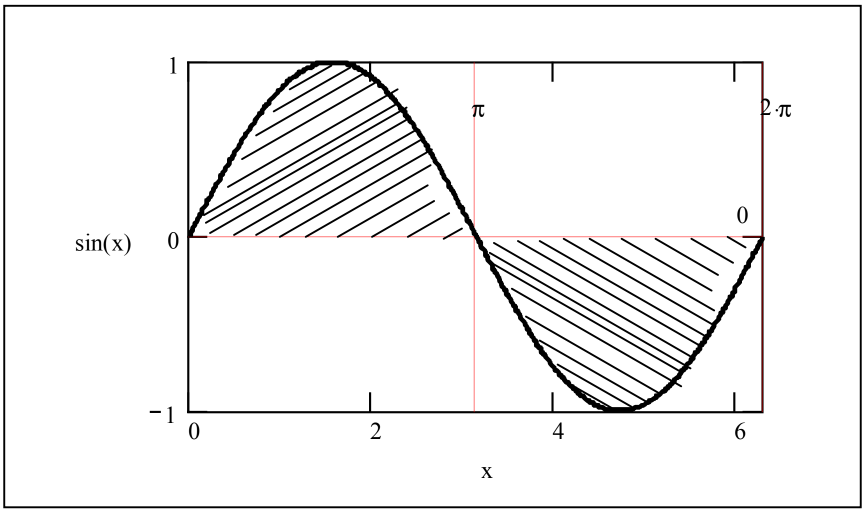
The units of $^{dy}/_{dx}$ will be m/s ÷ s = m/s^2, ie acceleration.

The units of $\int y\,dx$ will be m/s × s = m, ie distance.

If an integral has both negative and positive parts to it, then they will add to give a smaller sum. For example consider a sine function integrated over one cycle.

The area consists of a positive half and an equal negative half.

Therefore, the integral of this function between x=0 and x=2π must be zero.



We can confirm this by integrating.

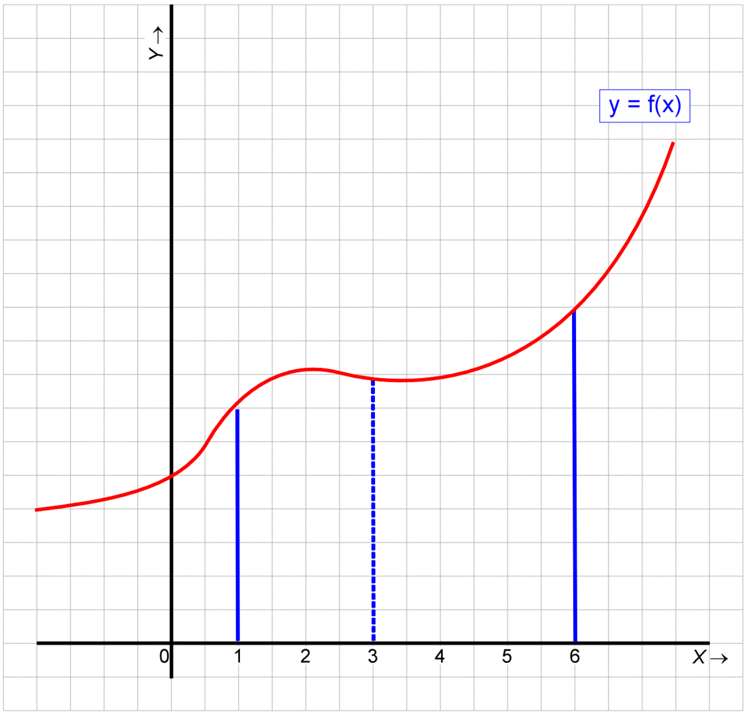
$$\int_0^{2\pi} \sin x \, \mathrm{d}x = [-\cos x]_0^{2\pi}$$

$$= -1 - (-1) = 0$$

Although the scalar magnitude of the "area" is not zero, the value of the integral *is* zero. Physical interpretations of negative integrals will be encountered later in your study of power in ac circuits.

It should be appreciated that "area" is merely a graphical interpretation of an integral, just as gradient is a graphical interpretation of a derivative. If the x and y axes were both calibrated in millimetres then the integral would of course be an actual area in mm^2. In an electrical problem, an integral would mean some other physical quantity. For example, if we had current as a function of time, then $\int i \, \mathrm{d}t$ would represent charge, and the unit would be amperes × seconds = coulombs.

Partitioning of an integral



p362 fig26

Consider the function $y = \mathrm{f}(x)$ as shown in the diagram. It is evident that the area under the curve between x=1 and x=6 is the same as the sum of the 2 areas from x=1 to x=3 and from x=3 to x=6

ie $$\int_1^6 \mathrm{f}(x)\,\mathrm{d}x = \int_1^3 \mathrm{f}(x)\,\mathrm{d}x + \int_3^6 \mathrm{f}(x)\,\mathrm{d}x$$

In general, we can say that if f(x) is integrable over the interval x=a to x=b, and point c lies in that interval, then

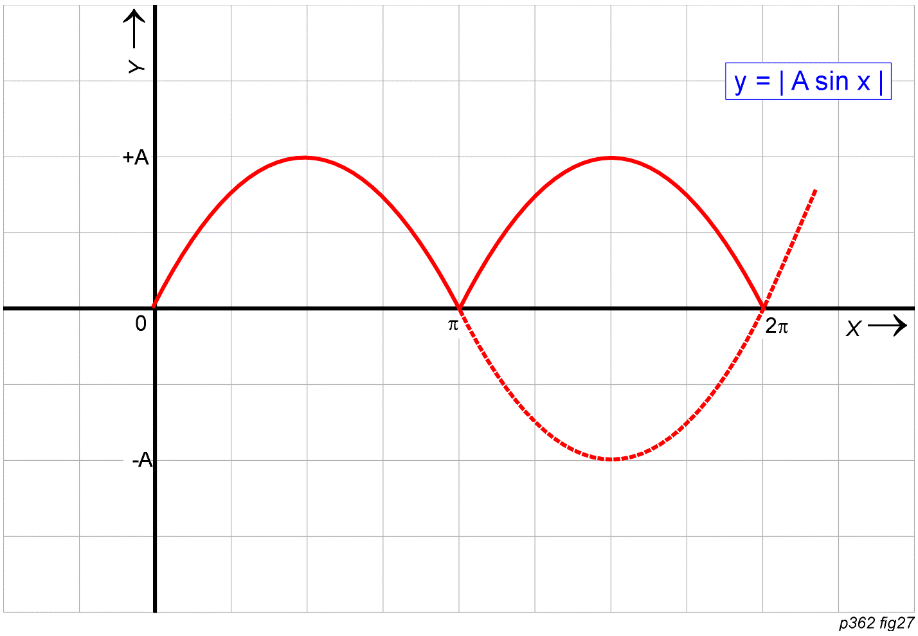
$$\int_a^b \mathrm{f}(x)\,\mathrm{d}x = \int_a^c \mathrm{f}(x)\,\mathrm{d}x + \int_c^b \mathrm{f}(x)\,\mathrm{d}x$$

It is sometimes necessary to partition a function into 2 or more parts in order to find the integral. This is because in some cases we cannot find a single expression for the indefinite integral over the whole interval.

There are many functions for which it is not actually possible to find an indefinite integral at all, although the definite integral exists and can be found by other means.

The following example, which is important in electrical theory, illustrates where partitioning is necessary.

Example



For example, consider a rectified sine wave of peak value A, represented by the solid line on the graph above.

This can be written as $y = |A \sin x|$
(The vertical lines mean the modulus or absolute value.)

To integrate this we have to split it into the sum of 2 functions. ie

$$y = \begin{cases} A \sin x \text{ for } 0 < x < \pi \\ -A \sin x \text{ for } \pi < x < 2\pi \end{cases}$$

Between x=0 and x=π, the function has the equation $y = A \sin x$.
Between x=π and x=2π the sine wave (shown dotted) has been inverted and therefore has the equation $y = -A \sin x$ over this interval.

Hence,

$$\int_0^{2\pi} |A \sin x|\, dx = \int_0^{\pi} A \sin x\, dx + \int_{\pi}^{2\pi} -A \sin x\, dx$$

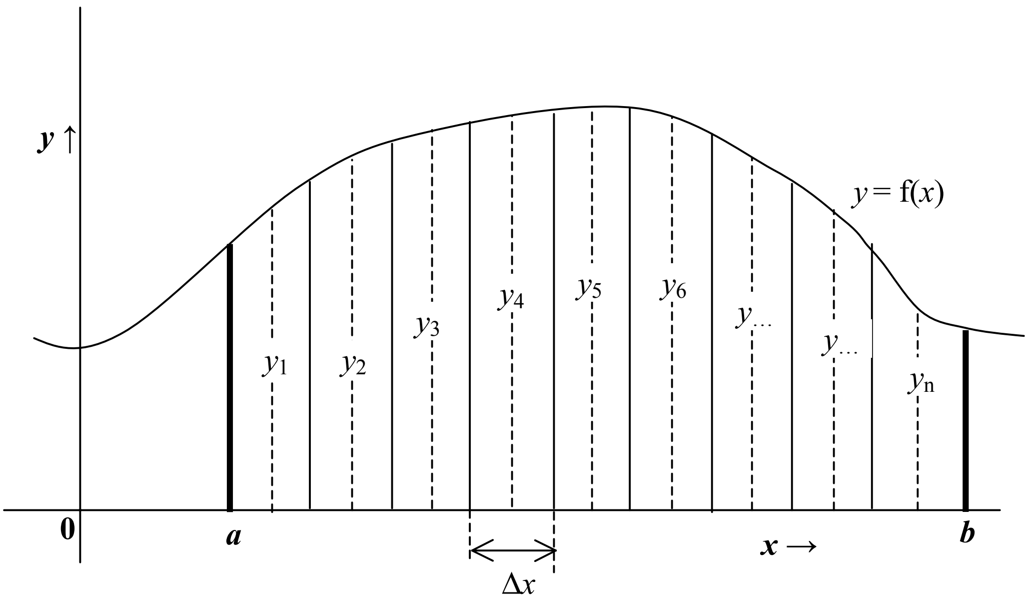
$$= A[-\cos x]_0^{\pi} + A[\cos x]_{\pi}^{2\pi}$$

$$= A\{-(-1)-(-1)\} + A\{1-(-1)\}$$

$$= 4A$$

The reason there is no sikngle expression for the indefinite integral over the whole interval is that, although function is continuous over the interval, the **derivative** is **discontinuous** within the interval.
For $0 < x < \pi$, ${}^{dy}/_{dx} = A \cos x$. For $\pi < x < 2\pi$, ${}^{dy}/_{dx} = -A \cos x$.
At x=π, **the derivative cannot be defined,** as it is different immediately to the left and right of this point. A function cannot be differentiated at a "sharp corner".

Mean values



Consider the function $y = \text{f}(x)$ in the interval $x=a$ to $x=b$.

If we divide the interval into n equal narrow strips, the mean value of y in that interval could be approximated by adding up the heights of all the mid–ordinates (the dotted lines) and divided by the number of ordinates, to give us the average or mean height. This is making the assumption that each small section of curve is a straight line and so each strip of width Δx is a trapezium. This method is quite accurate if the strips are narrow.

ie
Mean value of $y \cong (y_1 + y_2 + y_3 + \cdots + y_n \div n$

Which we can write more succinctly as: $1/n \sum_{i-1}^{n} y_i$

Now the area is the sum of the areas of all the trapeziums

which would be equal to $\sum_{i-1}^{n} y_i \;\Delta x \quad = \quad \Delta x \sum_{i-1}^{n} y_i$

The length of the interval is $(b–a)$ and so $\Delta x = (b–a) \div n$

$$\therefore \text{ area} \quad = \quad (b–a)\, 1/n \sum_{i-1}^{n} y_i \quad = \quad (b–a) \times \text{mean value}$$

This is what we would expect, ie the area under the curve is equal to the average height multiplied by the width.
Now, the values of y are continuously varying in the interval, and the true mean value and the true area would be obtained if we let Δx approach zero and n approach ∞, so that we are summing a greater number of narrower trapeziums.

In the limit, the area becomes $\int_a^b y \, \mathrm{d}x$

$\therefore$ $(b–a) \times$ mean value = $\int_a^b y \, \mathrm{d}x$

Hence mean value = $\dfrac{1}{b-a} \int_a^b y \, \mathrm{d}x$

Thus, if we know the function and can integrate it, we can find its mean value.

The rule is highlighted below.

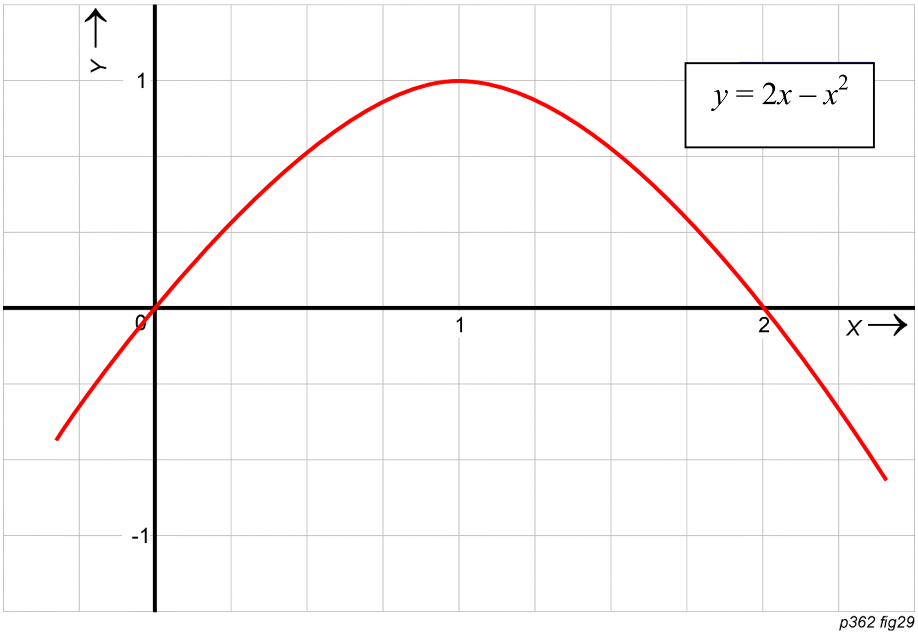
Rule for finding mean value

Mean value of f(x)
in the interval $x = a$ to $x = b$
is

$$\frac{1}{b-a} \int_a^b \mathrm{f}(x) \, \mathrm{d}x$$

i.e. integrate the function over the interval and divide by the length of the interval.

Example



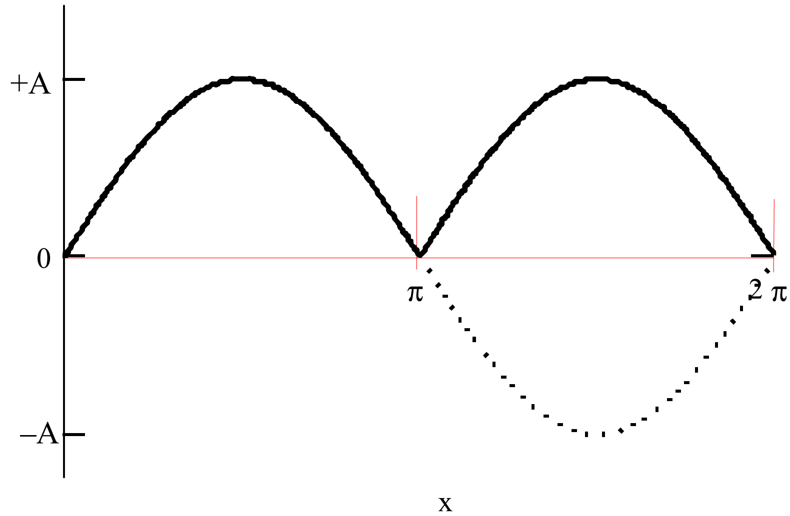
Mean value $= \dfrac{1}{2-0} \displaystyle\int_0^2 (2x - x^2)\,dx$

$= ½\,[x^2 - {}^1/_3\, x^3]_0^2$

$= ½\,(4 - {}^8/_3 - 0) \quad = \quad {}^2/_3$

Example

Find the mean value of the rectified sine wave $y = |A \sin x|$ in the interval x=0 to x=2π



From *page* 10, the value of the integral over this interval is 4A.

Therefore, the mean value of $4a \div 2\pi \quad = \quad 2A/\pi \quad \cong \quad 0{\cdot}637\,A$

You may recognise this figure from the theory of rectifiers.

SAQ3-7-2

Find the area between the x axis and the curve $y = 2x^3 - 12x^2 + 12x - 3$ from x=1 to x=4.
What does the negative answer indicate?

SAQ3-7-3

Evaluate $\int_0^{\pi} |\cos x|\, \mathrm{d}x$

SAQ3-7-4

Find the mean value of the function $f(x)=x^3$ in the interval x=1 to x=3.

Chapter 8

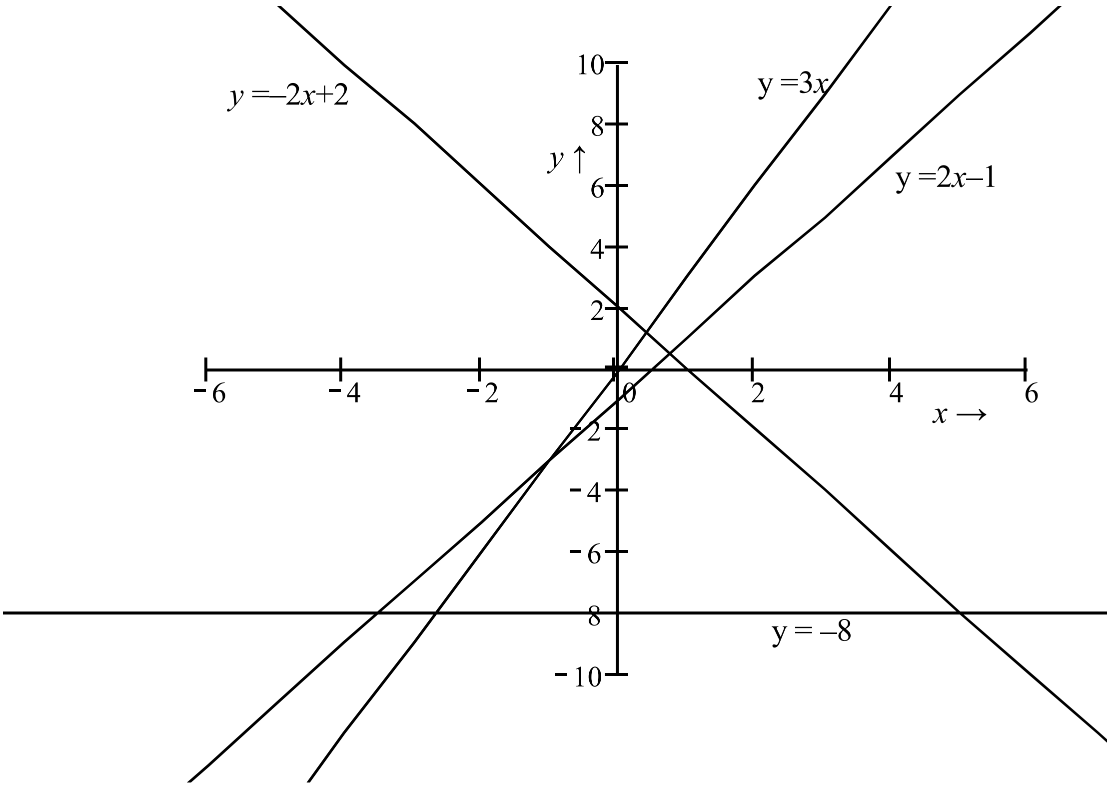
Solutions to SAQs

Solutions to SAQs

SAQ3-1-1

Equation	Gradient	Intercept
$y = 5x + 1$	5	1
$y = -2x + 4$	–2	4
$y = -1{\cdot}5x - 2$	–1·5	–2
$y = x - 3$	1	–3
$y = 4$	0	4

SAQ3-1-2



a. $y = 2x - 1$ Gradient = 2

b. $y = -2x + 2$ Gradient = –2

c. $y = 3x$ Gradient = 3

d. $y = -8$ Gradient = 0

SAQ3-1-3

a. $m = \frac{20-6}{3-1} = 7 \quad \therefore y = 7x + c$

substituting $x = 1$, $y = 6$; $6 = 7 + c$ $\therefore c = -1$

Equation is $y = 7x - 1$

b. $m = \frac{25+2}{4+5} = 3 \quad \therefore y = 3x + c$

substituting $x = -5$, $y = -2$; $-2 = 15 + c$ $\therefore c = 13$

Equation is $y = 3x + 13$

c. $m = \frac{5-15}{3+2} = -2 \quad \therefore y = -2x + c$

substituting $x = 3$, $y = 5$ $5 = -6 + c$ $\therefore c = 11$

Equation is $y = -2x + 11$

d. $m = \frac{4-20}{5-1} = -4 \quad \therefore y = -4x + c$

substituting $x = 1$, $y = 20$ $20 = -4 + c$ $\therefore c = 24$

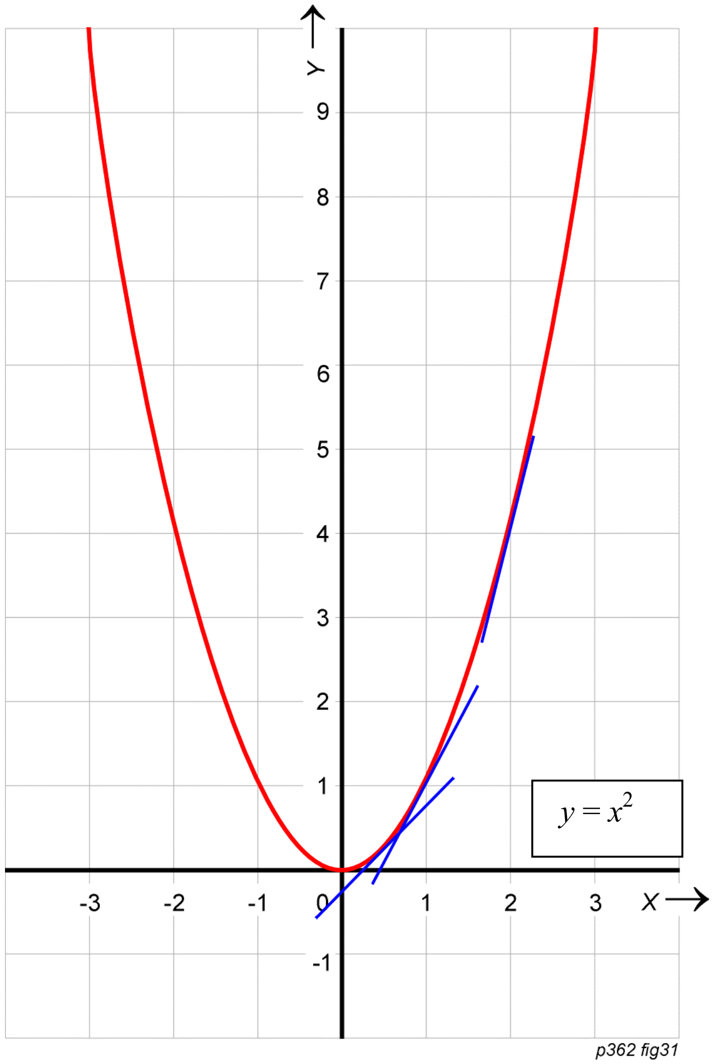
Equation is $y = -4x + 24$

e. $m = \frac{12+4}{6+2} = 2 \quad \therefore y = 2x + c$

substituting $x = -2$, $y = -4$ $-4 = -4 + x$ $\therefore c = 0$

Equation is $y = 2x$

SAQ3-1-4

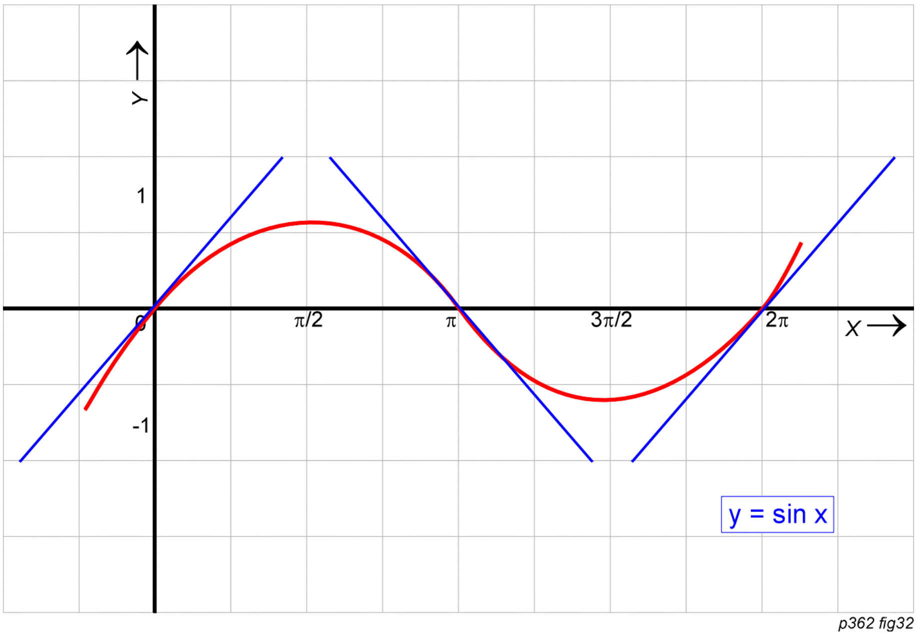


a. At $x = 1$, gradient = 2

b. At $x = 0{\cdot}5$, gradient = 1

c. At $x = 2$, gradient = 4

SAQ3-1-5



a. At $x = 0$, gradient = 1

b. At $x = \pi$, gradient = –1

c. At $x = 2\pi$, gradient = 1

At $x = \pi/2$ and $3\pi/2$, The slope is zero, as the tangents are horizontal.

SAQ3-2-1

By definition $$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

If $f(x) = x^3$

then $$f'(x) = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^3 - x^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\ \Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2 \Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 3x^2 + 3x\Delta x + (\Delta x)^2$$

$$= 3x^2$$

b. $f(2) = 2^3 = 8$, $f'(2) = 3 \times 2^2 = 12$

SAQ3-2-2

	y	${}^{dy}/_{dx}$
a.	x^5	$5x^4$
b.	x^{-2}	$-2x^{-3}$
c.	$x^{-1/2}$	$-\frac{1}{2}x^{-3/2}$
d.	$x^{3/2}$	${}^{3}/_{2}x^{1/2}$
e.	4	0

SAQ3-2-3

a. $y = 2x^2 - x + 2$

$dy/dx = 4x - 1$

b. $y = 4x^5 + 2x^3 - 5x^2 - 3x - 7$

$dy/dx = 20x^4 + 6x^2 - 10x - 3$

c. $y = 1/x^3 = x^{-3}$

$dy/dx = -3x^{-4} = -3/x^4$

d. $y = \sqrt{x} = x^{1/2}$

$dy/dx = \frac{1}{2}x^{-1/2} = 1/(2\sqrt{x})$

e. $y = 2\sqrt{x} - \sqrt{(x^3)} = 2x^{1/2} - x^{3/2}$

$dy/dx = x^{-1/2} - {}^{3}/{}_{2}x^{1/2} = 1/\sqrt{x} - {}^{3}/{}_{2}\sqrt{x}$

f. $y = (x - 3)^2 = x^2 - 6x + 9$

$dy/dx = 2x - 6$

g. $y = x^2 - 1/x^2 = x^2 - x^{-2}$

$dy/dx = 2x + 2x^{-3} = 2x + 2/x^3$

h. $y = \ln(x^2) = 2 \ln x$

$dy/dx = 2/x$

SAQ3-2-4

$$f(x) = 2x^3 - 9x^2 + 10$$

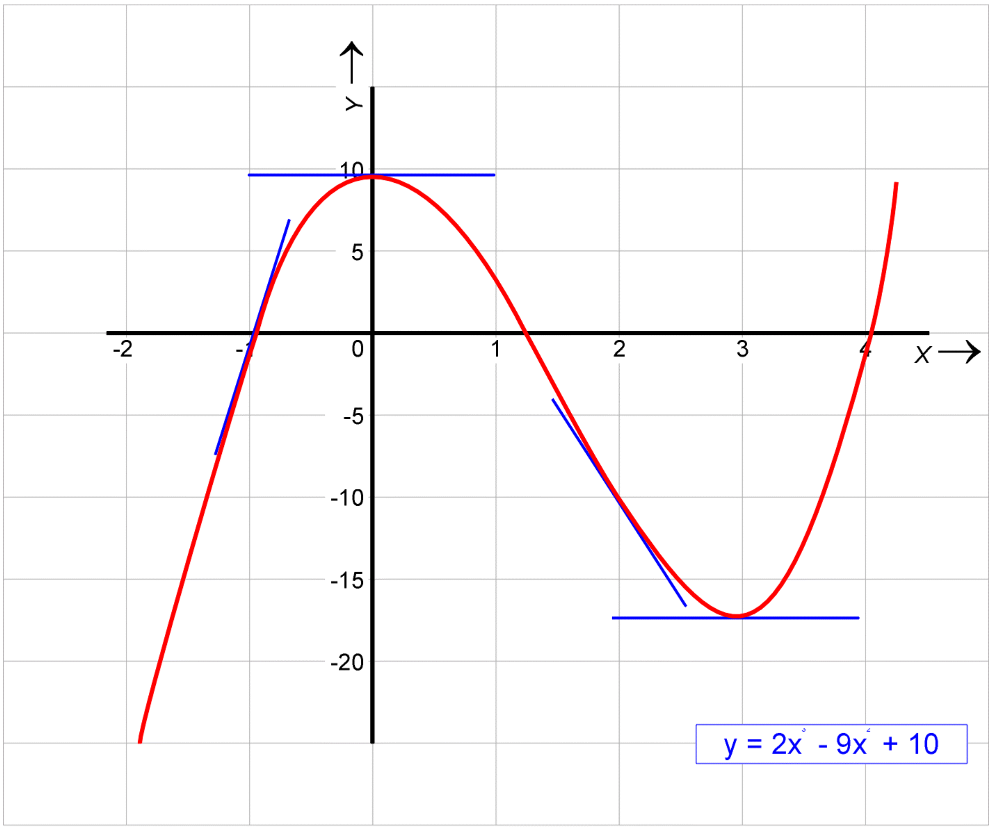
$$f'(x) = 6x^2 - 18x$$

a. $f'(2) = -12$

b. $f'(0) = 0$

c. $f'(3) = 0$

d. $f'(-1) = 24$



p362 fig33

SAQ3-3-1

a. $y = 3x^2 \tan x$

$^{dy}/_{dx} = 3x^2 \sec^2 x + 6x \tan x$

b. $y = x \ln x - x$

$^{dy}/_{dx} = x/x + \ln x - 1 = \ln x$

c. $y = e^x \sin x \cos x$

Let $u = e^x \sin x$, $v = \cos x$

then $^{du}/_{dx} = e^x \cos x + e^x \sin x$

$= e^x(\cos x + \sin x)$

$^{dv}/_{dx} = -\sin x$

$^{dy}/_{dx} = e^x \sin x\,(-\sin x) + \cos x\, e^x(\cos x + \sin x)$

$= e^x(\cos^2 x - \sin^2 x + \cos x \sin x)$

SAQ3-3-2

a. $$y = \frac{x^3 - x^2 + 3}{2x+1}$$

$$\frac{dy}{dx} = \frac{(2x+1)(3x^2 - 2x) - (x^3 - x^2 + 3)2}{(2x+1)^2}$$

$$= \frac{4x^3 + x^2 - 2x - 6}{(2x+1)^2}$$

b. $$y = \frac{x^2}{3x+5}$$

$$\frac{dy}{dx} = \frac{(3x+5)3x - 3x^2}{(3x+5)^2}$$

$$= \frac{3x^2 + 10x}{(3x+5)^2}$$

c. $$y = \frac{5x^2 e^x}{1+x^2}$$

Let $u = 5x^2 e^x$, $v = 1 + x^2$

Then ${}^{du}/_{dx} = 5x^2 e^x + 10x e^x = 5xe^x(x + 2)$

${}^{dv}/_{dx} = 2x$

$$\frac{dy}{dx} = \frac{(1+x^2)\, 5xe^x(x+2) - (2x)5x^2 e^x}{(1+x^2)^2}$$

$$= \frac{5xe^x(x^3 + x + 2)}{(1+x^2)^2}$$

SAQ3-4-1

a. $y = \sqrt{2x^2 + 4x}$

Let $z = 2x^2 + 4x$, $y = z^{½}$
$^{dz}/_{dx} = 4x + 4$, $^{dy}/_{dz} = ½z^{-½} = ½(2x^2 + 4x)^{-½}$

$$^{dy}/_{dx} = {}^{dy}/_{dz}\ {}^{dz}/_{dx} = (2x + 2)(2x^2 + 4x)^{-½}$$

$$= \frac{2(x+1)}{\sqrt{2x^2+4x}}$$

b. $y = \tan^2 x$

Let $z = \tan x$, $y = z^2$
$^{dz}/_{dx} = \sec^2 x$, $^{dy}/_{dz} = 2z = 2\tan x$

$$^{dy}/_{dx} = {}^{dy}/_{dz}\ {}^{dz}/_{dx} = 2\tan x\sec^2 x$$

c. $y = \ln(x^3 + 3x)$

Let $z = x^3 + 3x$, $y = \ln z$
$^{dz}/_{dx} = 3x^2 + 3$, $^{dy}/_{dz} = 1/z = 1/(x^3 + 3x)$

$$^{dy}/_{dx} = {}^{dy}/_{dz}\ {}^{dz}/_{dx} = \frac{3x^2+3}{x^3+3x}$$

d. $y = e^{2x+1}$
Let $z = 2x + 1$, $y = e^z$
$^{dz}/_{dx} = 2$, $^{dy}/_{dz} = e^z = e^{2x+1}$

$$^{dy}/_{dx} = {}^{dy}/_{dz}\ {}^{dz}/_{dx} = 2e^{2x+1}$$

e. $y = \ln(\sec x + \tan x)$

Let $z = \sec x + \tan x$, $y = \ln z$

$^{dz}/_{dx} = \sec x\tan x + \sec^2 x$, $^{dy}/_{dz} = 1/z = 1/(\sec x + \tan x)$

$$^{dy}/_{dx} = {}^{dy}/_{dz}\ {}^{dz}/_{dx} = \frac{\sec x\tan x + \sec^2 x}{\sec x + \tan x}$$

$$= \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} = \sec x$$

f. $y = (3x+2)^{12}$

Let $z = 3x + 2$, $y = z^{12}$

${}^{dz}/_{dx} = 3$, ${}^{dy}/_{dz} = 12z^{11} = (3x+2)^{11}$

${}^{dy}/_{dx} = {}^{dy}/_{dz}\ {}^{dz}/_{dx} = 3\,(3x+2)^{11}$

g. $y = \sin(2x + \pi/6)$

Let $z = 2x + \pi/6$, $y = \sin z$

${}^{dz}/_{dx} = 2$, ${}^{dy}/_{dz} = \cos z = \cos(2x + \pi/6)$

${}^{dy}/_{dx} = {}^{dy}/_{dz}\ {}^{dz}/_{dx} = 2\cos(2x + \pi/6)$

SAQ3-4-2

$y = \ln(\cos 4x)$

Let $v = 4x$, $u = \cos v$, $y = \ln u$

${}^{dv}/_{dx} = 4$, ${}^{du}/_{dv} = -\sin v = -\sin 4x$, ${}^{dy}/_{du} = 1/u = 1/\cos v = 1/\cos 4x$

$${}^{dy}/_{dx} = {}^{dy}/_{du}\ {}^{du}/_{dv}\ {}^{dv}/_{dx}$$

$$= \frac{-4\sin 4x}{\cos 4x} = -4\tan 4x$$

SAQ3-4-3

$$\cos x = \tfrac{1}{2}(e^{-jx})$$

$$^{d}/_{dx}(\cos x) = \tfrac{1}{2}(je^{jx} - je^{-jx})$$

$$= \frac{j(e^{jx} - e^{-jx})}{2}$$

$$= \frac{j^2(e^{jx} - e^{-jx})}{2j}$$

$$= \frac{-(e^{jx} - e^{-jx})}{2j} = -\sin x$$

SAQ3-5-1

	$y = f(x)$	=	$x^4 + 4x^3 + 2x^2 - 2x + 1$		
a.	$^{dy}/_{dx}$	=	$4x^3 + 12x^2 + 4x - 2$		
b.	$^{d^2y}/_{dx^2}$	=	$12x^2 + 24x + 4$		
c.	$^{d^3y}/_{dx^3}$	=	$24x + 24$		
d.	$f'(2)$	=	$4\times2^3 + 12\times2^2 + 4\times2 - 2$	=	86
e.	$f''(1)$	=	$12\times1^2 + 24\times1 + 4$	=	40
f.	$f'''(-1)$	=	$24\times(-1) + 24$	=	0

SAQ3–5–2

$s = 5t^4 - 14t^3 + 8t^2$

a. We have to find the values of t when s=0.

i.e. when $5t^4 - 14t^3 + 8t^2 = 0$

$t^2(5t^2 - 14t + 8) = 0$

This gives the solutions; t^2=0, $\therefore$ t=0, which is the starting time, and $5t^2 - 14t + 8 = 0$ which is a quadratic equation with 2 roots.

We can find these by factorising, ie $(5t - 4)(t - 2) = 0$

$\therefore$ $t = 0{\cdot}8$ seconds, $t = 2$ seconds

$^{ds}/_{dt} = 20t^3 - 42t^2 + 16t$

$f'(0{\cdot}8) = -3{\cdot}84$, hence velocity at this point = –3·84 m/s

$f'(2) = 24$, hence velocity at this point = 24 m/s

$^{d^2s}/_{dt^2} = 60t^2 - 84t + 16$

$f''(0{\cdot}8) = -12{\cdot}8$, hence acceleration at this point = –12·8 m/s^2

$f''(2) = 88$, hence acceleration at this point = 88 m/s^2

b. When velocity is zero, $20t^3 - 42t^2 + 16t = 0$

$\therefore$ $2t(10t^2 - 21t + 8) = 0$

$2t(2t - 1)(5t - 8) = 0$

This gives the solutions; t=0, 0·5, 1·6 seconds.

At t=0, $f''(0) = 16$, hence acceleration at this point = 16 m/s^2

At t=0·5, $f''(0{\cdot}5) = -11$, hence acceleration at this point = –11 m/s^2

At t=1·6, $f''(1{\cdot}6) = 35{\cdot}2$, hence acceleration at this point = 35·2 m/s^2

SAQ3-6-1

f(*x*)	$\int f(x)dx$
x^5	$^1/_6\, x^6$ + **C**
$\sqrt{x} = x^{1/2}$	$^2/_3\, x^{3/2}$ + **C**
$1/\sqrt{x} = x^{-1/2}$	$2x^{1/2}$ + **C** = $2\sqrt{x}$ + **C**
x^{-2}	$-x^{-1}$ + **C** = $-1/x$ + **C**
x	$\frac{1}{2}x^2$ + **C**
3	$3x$ + **C**

SAQ3-6-2

a. $\int (3x^5 + 8x^3 - 15x^2 + x = 1)\mathrm{d}x$

$= \tfrac{1}{2}x^6 + 2x^4 - 5x^3 + \circ x^2 - x + \mathbf{C}$

b. $\int \frac{2}{x}\mathrm{d}x = 2\int x^{-1}\mathrm{d}x$

$= 2\ln x + \mathbf{C} = \ln(\mathbf{K}x^2)$

c. $\int \frac{dx}{2\sqrt{x}} = \tfrac{1}{2}\int x^{-\frac{1}{2}}\,\mathrm{d}x$

$= \tfrac{1}{2} \times 2x^{\frac{1}{2}} + \mathbf{C} = \sqrt{x} + \mathbf{C}$

d. $\int \frac{2}{x^3}\mathrm{d}x = 2\int x^{-3}\,\mathrm{d}x$

$= -x^{-2} + \mathbf{C} = -1/x^2 = \mathbf{C}$

e. $\int (3\cos x + 2\sin x)\,\mathrm{d}x$

$= 3\int \cos x\,\mathrm{d}x + 2\int \sin x\,\mathrm{d}x = 3\sin x - 2\cos x + \mathbf{C}$

f. $\int (x + {}^{1}/_{\sqrt{x}})\,\mathrm{d}x$

$= \int x\,\mathrm{d}x + \int x^{-\frac{1}{2}}\,\mathrm{d}x = \tfrac{1}{2}x^2 + 2x^{\frac{1}{2}} + \mathbf{C}$

$= \tfrac{1}{2}x^2 + 2\sqrt{x} + \mathbf{C}$

SAQ3-6-3

a. $\int e^{2x}\,dx$

$= \tfrac{1}{2}e^{2x} + \mathbf{C}$

b. $\int e^{3x-2}\,dx$

$= {}^{1}/_{3}\, e^{3x-2} + \mathbf{C}$

c. $\int e^{-x}\,dx$

$= -e^{-x} + \mathbf{C}$

d. $\int \cos\left(2x + \pi/3\right) dx$

$= \tfrac{1}{2}\sin(2x + \pi/3) + \mathbf{C}$

e. $\int \sin\left(0\cdot 016t + 0\cdot 5\right) dt$

$= -100\cos(0\cdot 01t + 0\cdot 5) + \mathbf{C}$

SAQ3-7-1

a. $\int_1^3 (10x^3 - 2x^2 + 6x - 1)\mathrm{d}x$

$= [{}^5/_2x^4 - {}^2/_3x^3 + 3x^2 - x]_1^3$

$= ({}^5/_2\times81 - {}^2/_3\times27 + 3\times9 - 3) - {}^5/_2 - {}^2/_3 + 3 - 1)$

$= 204\,{}^2/_3$

b. $\int_0^{\pi/6} 2 \cos x \, \mathrm{d}x$

$= [2 \sin x]_0^{\pi/6}$

$= 2(½ - 0)$

$= 1$

c. $\int_1^4 3\sqrt{x}\, \mathrm{d}x$

$= 3\int_1^4 x^{1/2}\, \mathrm{d}x$

$= [2x^{3/2}]_1^4$

$= 2\times8 - 2\times1$

$= 14$

SAQ3-7-2

$$\int_1^4 \left(2x^3 - 12x^2 + 12x - 3\right) \mathrm{d}\,x$$

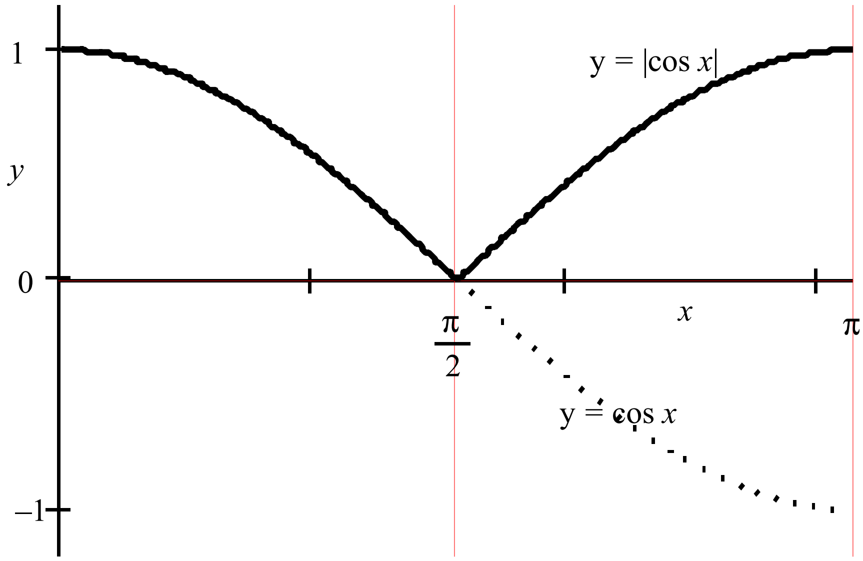
$$= \quad [\tfrac{1}{2}x^4 - 4x^3 + 6x^2 - 3x]_1^4$$

$$= \quad (\tfrac{1}{2}\times256 - 4\times64 + 6\times16 - 3\times4) - (\tfrac{1}{2} - 4 + 6 - 3)$$

$$= \quad -43\tfrac{1}{2}$$

The negative answer indicates that the area or most of the area lies below the x axis.

SAQ3-7-3



It can be seen that the function $y = |\cos x|$ has a discontinuous derivative at $x = \pi/2$. To integrate it, we have to split it into 2 parts.

$$\int_0^{\pi} \left|\cos x\right| \mathrm{d}\,x \quad = \quad \int_0^{\pi/2} \cos x \,\mathrm{d}\,x \quad + \quad \int_{\pi/2}^{\pi} -\cos x \,\mathrm{d}\,x$$

$$= \quad [\sin x]_0^{\pi/2}$$

$$= \quad (1 - 0) - (0 - 1)$$

$$= \quad 2$$

SAQ3-7-4

Mean value $= \frac{1}{3-1}\int_1^3 x^3 \mathrm{d}x$

$= \frac{1}{2}\left[\frac{1}{4}x^4\right]_1^3$

$= \frac{1}{8}(81 - 1)$

$= 10$