THE ROYAL SCHOOL OF SIGNALS

TRAINING PAMPHLET NO: **361**

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MODULE 2 – COMPLEX NUMBERS

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C O N T E N T S

Chapter 1

Real, imaginary and complex numbers

Imaginary numbers

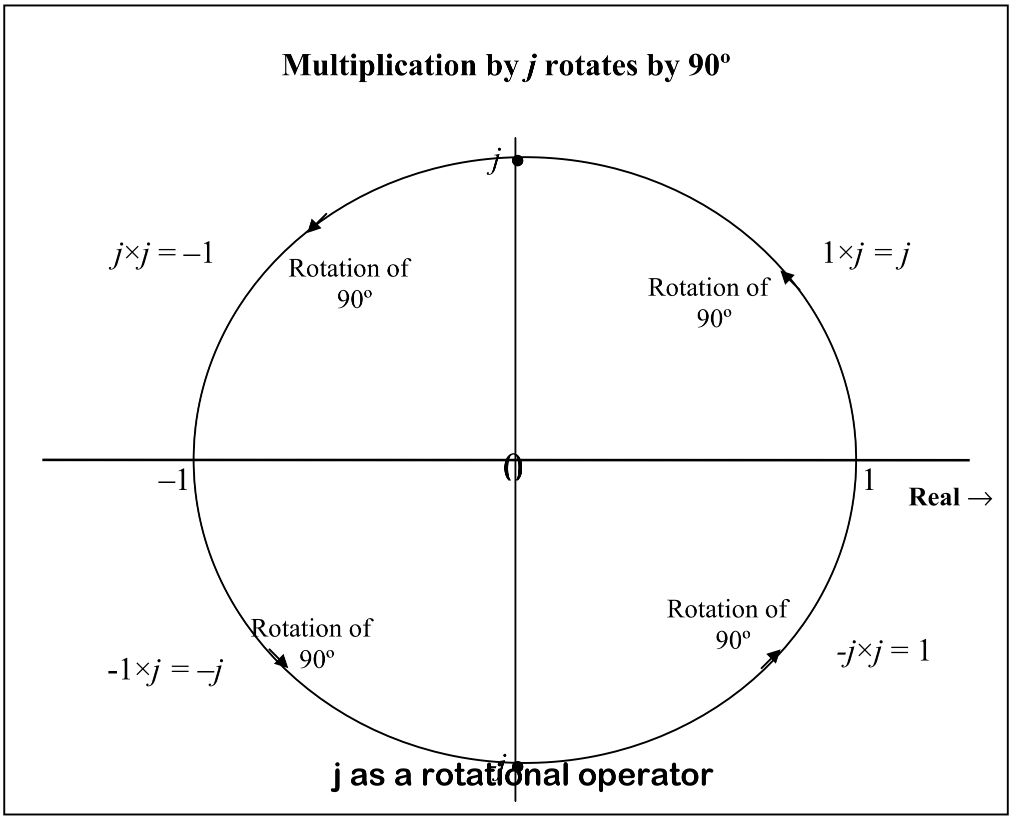
We saw in Section 1, chapter 1 that the *Real numbers* consist of rational and irrational numbers and can be represented graphically by points on a line. We also saw that certain equations have no solution amongst the real numbers.
For example $x^2 = -1$ has no *real* solution since multiplying any number, positive or negative, by itself gives a positive result. In order to provide solutions to such problems, the number system was extended and the so-called *imaginary numbers* were conceived.

We define a number j such that $j^2 = -1$. Note that in pure mathematics texts, *i* is used. In electrical engineering we use j so as not to cause confusion with the symbol for current. j is called an imaginary number

The term "imaginary" is perhaps an unfortunate one since it implies that imaginary numbers have no actual meaning. However, all numbers such as negative numbers and irrational numbers were originally an extension of the number system, necessary for the solution of new problems, and were therefore "imagined" by someone. We are all perfectly familiar with everyday applications of fractions and negative numbers,. and as we shall see, imaginary numbers also have practical physical interpretations.

The j operator

It is obvious that j does not fit anywhere on our Real Line. You will recall from Section 1 that multiplication by –1 gives a rotation of 180º. Since $j^2 = -1$, it seems reasonable to assume that a multiplication by j gives a rotation of 90º. Multiplication by j again, ie by $j^2 = -1$ gives a further 90º rotation to 180º, bringing us back on to the Real Line at –1.



Multiplying –1 by j gives a further 90º rotation indicating that multiplying 1 by –j would give a rotation of 90º clockwise ie –90º. Multiplying –j by j again brings us back to 1.

Thus we can see that

$1 \times j$	$= j$	rotation of 90º from 1
$j \times j$	$= -1$	rotation of 180º from 1
$-1 \times j$	$= -j$	rotation of 270º or –90º from 1
$-j \times j$	$= 1$	rotation of 360° or 0º from 1

Thus the imaginary number j may be regarded as a *rotational operator.* This has useful applications to AC circuit theory where a quantity such as a voltage which is in *quadrature* may be represented as multiplied by j to give a rotation of 90º or multiplied by –j to give a rotation of –90º.

SAQ2-1-1

Write down the values of:

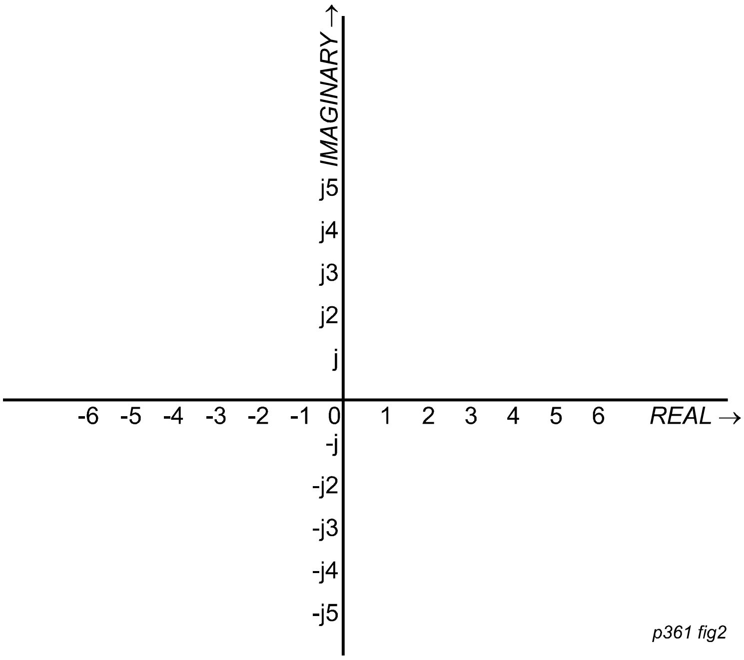
a. j^2　　b. j^3　　c. j^4

d. $-j^2$　　e. $(-j)^2$　　f. j^5

g. j^6　　h. $(-j)^4$　　i. $-j^4$

The Argand diagram

Since multiplying a real number by j represents a rotation of 90º, we may represent the imaginary numbers graphically as lying on an axis at right angles to the real line.



This graphical representation is called an Argand diagram.

The imaginary numbers lie on the *imaginary axis* at 90º to the *real axis,* so for example, the imaginary number j3 would be obtained by rotating 90º from the real, number 3. The imaginary number –j4 would be obtained by rotating 90º from the real number –4 or by rotating –90º from the real number +4.

It does not matter whether we write, for example, j3 or 3j. They are the same thing. In electrical engineering, where we regard j as a rotational operator, we tend to write it in the form j3, implying that it is the real quantity 3 rotated by 90º, ie in quadrature. In mathematics texts, they tend to write it in the form 3j or more commonly, 3*i*.

Note that the real and imaginary numbers do not coincide at any point other than zero.

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SAQ2-1-2

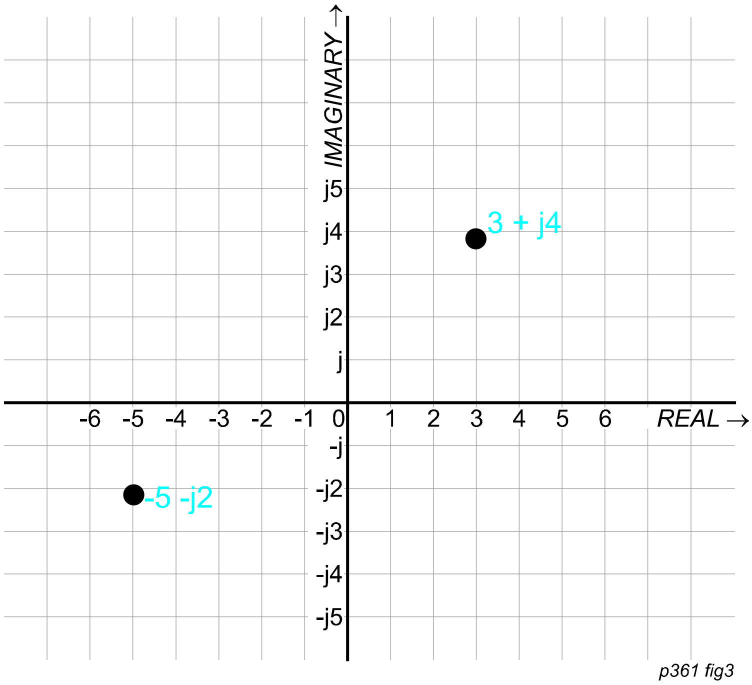
Simplify

a.	$5 \times j$	b.	$6 \times -j$	c.	$j2 \times 3$
d.	$j4 \times j$	e.	$-j2 \times -j$	f.	$-j2 \times j7$
g.	$(j2)^2$	h.	$(-j2)^3$	i.	$-(-j2)^4$

`

Complex numbers

A *complex* number is a number which contains a real part and an imaginary part, ie $z = a + jb$ is a **complex number,** where *a, b* are real numbers. The *real part* is *a* and the *imaginary part is* j*b*. On the Argand diagram a complex number is represented by a point in the plane.



For example, the point 3+j4 has a real coordinate of 3 and an imaginary coordinate of 4. The point –5–j2 has a real coordinate of –5 and an imaginary coordinate of –2. This plane is called the complex plane.

Rectangular form

Complex numbers written in the form of real and imaginary coodinates, ie the form $a + jb$, are said to be in *rectangular* form.

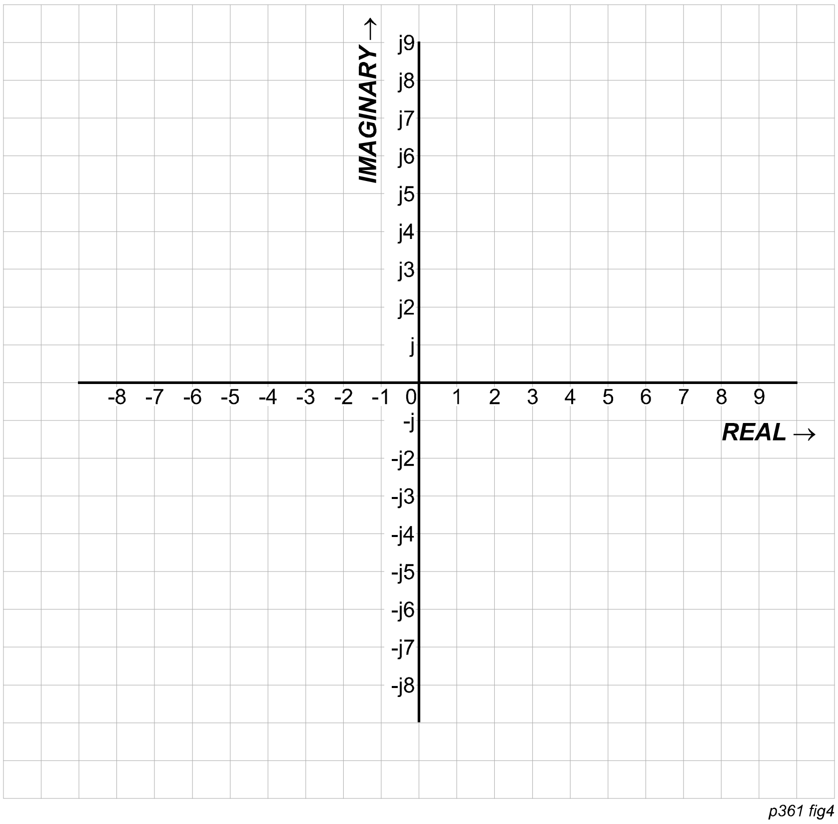
A real number may be regarded as a complex number with zero imaginary part. Hence, the real numbers are a subset of the complex numbers. Similarly a purely imaginary number may be regarded as a complex number with zero real part.

SAQ2-1-3

a. Write down the Real part and the imaginary part of the complex numbers in the table.

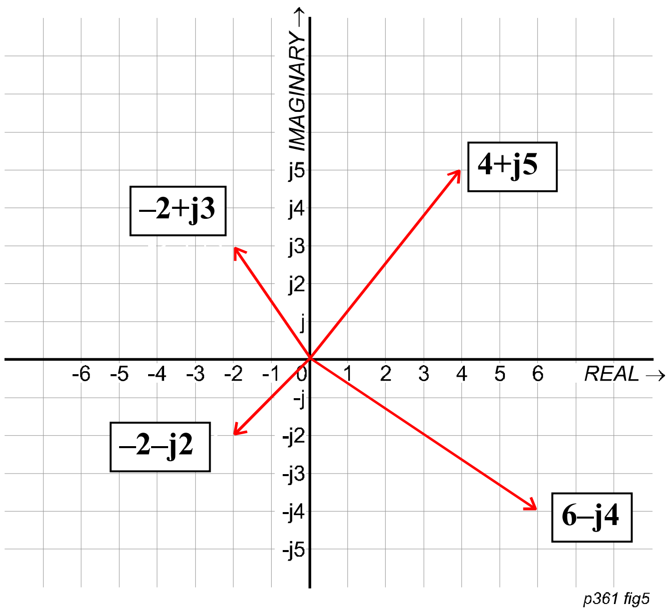
Number	Real part	Imaginary part
$5 + j4$		
$3 - j2$		
$-1 - j$		
6		
$j8$		
$\sqrt{9}$		
$\sqrt{-9}$		

b. Mark the complex numbers from the table in part (a), on the Argand diagram below.



Representation of vectors

One important application of complex numbers is their use to represent vectors. It is convenient to use the complex number a+jb to represent the vector joining the origin 0+j0 to the point a+jb. The vectors may then be added and multiplied using complex number arithmetic, which considerably simplifies their manipulation.



Representation of vectors by complex numbers

Chapter 2

Complex number arithmetic in rectangular form

Complex number arithmetic	The complex numbers follow the same basic laws as real numbers, ie If *U, V, W,* are complex numbers:

$$U + V = V + U$$
$$UV = VU \qquad \text{(commutative laws)}$$

$$(U + V) + W = U + (V + W)$$
$$(UV)W = U(VW) \qquad \text{(associative laws)}$$

$$U(V + W) = UV + UW \qquad \text{(distributive law)}$$

Addition and subtraction of complex numbers

The addition of complex numbers is similar to the addition of vectors, ie the horizontal and vertical components are added separately. To add 2 complex numbers, simply add their real parts and add their imaginary parts.

If $z_1 = a + \mathrm{j}b$, $z_2 = c + \mathrm{j}d$, then

$$z_1 + z_2 = a + c + \mathrm{j}b + \mathrm{j}d = (a + c) + \mathrm{j}(b + d)$$

Examples

$$2 + \mathrm{j}3 + 4 + \mathrm{j}5 = 6 + \mathrm{j}8$$

$$-5 + \mathrm{j}6 + 2 - \mathrm{j} = -3 + \mathrm{j}5$$

$$3 - \mathrm{j}2 + -1 + \mathrm{j}7 = 2 + \mathrm{j}5$$

Subtraction is similar:

If $z_1 = a + \mathrm{j}b$, $z_2 = c + \mathrm{j}d$, then

$$z_1 - z_2 = a - c + \mathrm{j}b - \mathrm{j}d = (a - c) + \mathrm{j}(b - d)$$

Examples

$$4 + \mathrm{j}2 - (3 + \mathrm{j}4) = 1 - \mathrm{j}2$$

$$-6 - \mathrm{j}5 - (3 - \mathrm{j}7) = -9 + \mathrm{j}2$$

SAQ2-2-1

For the following values of z_1 and z_2, calculate (i) $z_1 + z_2$ (ii) $z_1 - z_2$.

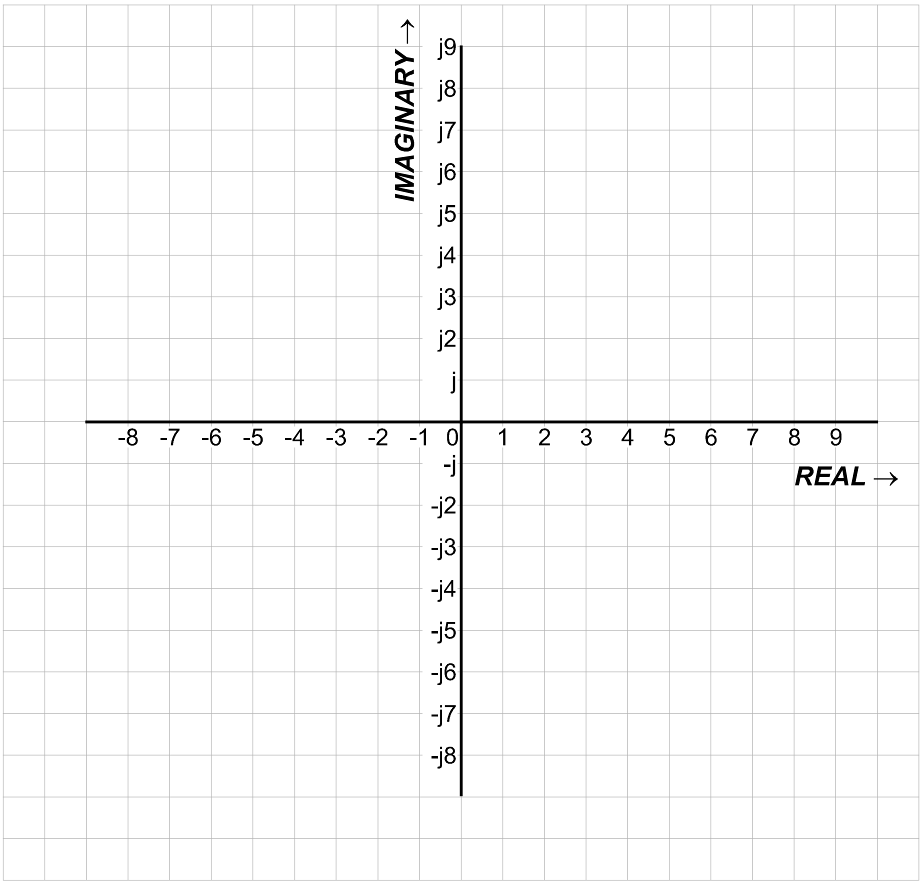
	z_1	z_2	$z_1 + z_2$	$z_1 - z_2$
a.	5 + j2	3 + j4		
b.	–3 + j	4 + j9		
c.	5 – j3	6 – j7		
d.	–3 + j2	8 – j10		
e.	–2 – j	–5 – j12		

SAQ2-2-2

If $z_1 = 2 + j6$, $z_2 = 5 - j2$

plot the following complex numbers as vectors on the Argand diagram below:

a. z_1 b. z_2 c. $-z_2$ d. $z_1 + z_2$ e. $z_1 - z_2$



p361 fig4

Do the rules for adding and subtracting complex numbers confirm the parallelogram rule for vectors? Sketch in the parallelograms and check.

This part has been left blank for working SAQs

Applications to AC networks

One of the most important uses of complex numbers is the representation of vector quantities in AC circuit theory. This section does not discuss AC theory which will be covered in **Section 5; Electrical Principles.** However, the student will probably be aware already, that a quantity such as a voltage rotated in phase by ±90º may be regarded as multiplied by ±j, and that an impedance is represented by a complex number whose real part is the resistance and whose imaginary part is the reactance, so that we can write the impedance of a series LCR circuit as

$$Z = R + \mathrm{j}(\omega L - 1/\omega C)$$

The various components of an AC network may then be represented by complex numbers and problems may be solved using complex number arithmetic. This considerably simplifies the solution of AC networks.

All problems in this section may be solved without any knowledge of electrical theory.

Use of calculators

Some scientific calculators will perform complex arithmetic. Initially, you should solve the SAQs without this facility, using the calculator for addition, subtraction, multiplication, division, and trigonometric functions only, in order to become familiar with the methods. Subsequently you could use the calculator to check your answers.

Multiplication of complex numbers	Complex numbers are multiplied together using the distributive law of multiplication, **remembering that** $j^2 = -1$.
	$$\begin{aligned}(a + jb)(c + jd) &= a(c + jd) + jb(c + jd) \\ &= ac + jad + jbc + j^2bd \\ &= ac + jad + jbc - bd \\ &= ac - bd + j(ad + bc)\end{aligned}$$
Examples	a. $(2 + j3)(4 + j5) = 8 + j10 + j12 + j^2 15$ $= 8 + j10 + j12 - 15 \quad = -7 + j22$
	b. $(3 - j7)(2 + j6) = 6 + j18 - j14 - j^2 42$ $= 6 + j18 - j14 + 42 \quad = 48 + j4$
	c. $(-2 - j8)(1 - j3) = -2 + j6 - j8 + j^2 24$ $= -2 + j6 - j8 - 24 \quad = -26 - j2$
	d. $j(7 + j5) = j7 + j^2 5$ $= j7 - 5 \quad = -5 + j7$
	e. $(1 + j)^2 = 1 + j^2 + j2$ $= 1 - 1 + j2 \quad = j2$
	f. $(1 + j)(1 - j) = 1 - j + j - j^2$ $= 1 + 1 \quad = 2$
	g. $({}^1/_{\sqrt{2}} + {}^j/_{\sqrt{2}})\,(-{}^1/_{\sqrt{2}} - {}^j/_{\sqrt{2}}) = -½ - j½ - j½ - j^2½$ $= -½ - j½ - j½ + ½ \quad = -j$

SAQ2-2-3

Calculate $z_1 z_2$ in the form $a + jb$ for the following complex numbers:

	z_1	z_2	$z_1 z_2$
a.	5 + j2	3 + j4	
b.	–3 + j7	6 + j8	
c.	–4 – j	5 + j2	
d.	12 + j7	9 – j	
e.	3 – j2	–4 – j5	
f.	–8 – j3	–3 – j5	
g.	$\frac{1}{2} + j\,\frac{\sqrt{3}}{2}$	$\frac{1}{2} + j\,\frac{\sqrt{3}}{2}$	

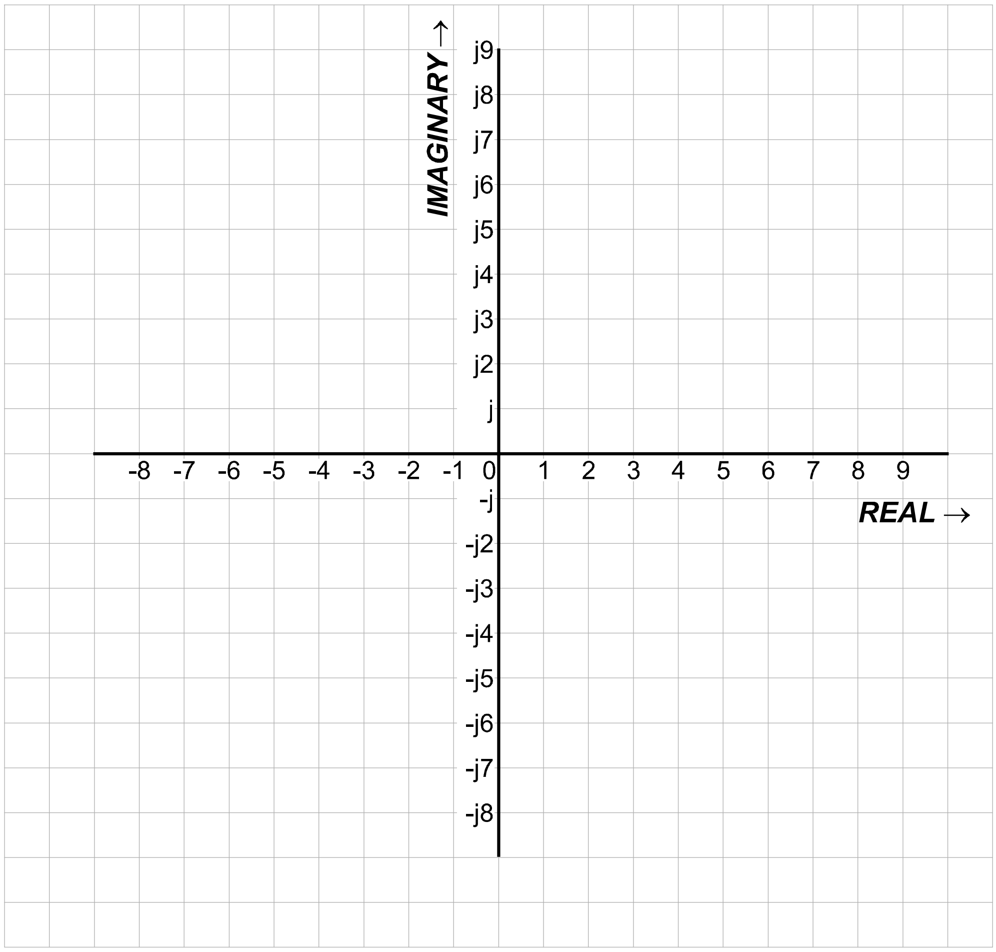
SAQ2-2-4

a. Plot on the Argand diagram below, the vector representing $z_1 = 5 + j7$

b. Calculate $z_2 = j(5 + j7)$ in the form $a + jb$ and plot this vector also.

c. Measure the angle between z_1 and z_2. What rule does the result confirm?

d. Calculate $z_3 = -j(5 + j7)$ in the form $a + jb$ and plot this vector also.



p361 fig4

What are the angles between (i) z_1 and z_3 (ii) z_2 and z_3?
What principle does this illustrate?

Complex conjugate

The **conjugate** of the complex number a + jb is the number a –jb. For example, the conjugate of 2+j3 is 2–j3. The conjugate of 4–j5 is 4+j5.

The conjugate of the complex number z if often denoted z^* or $\overline{z}$.

Rule: To find the conjugate of a complex number, change the sign of the imaginary part.

The complex conjugate is a very useful tool. The sum or product of a complex number and its conjugate are always real.

Sum: $(a + \mathrm{j}b) + (a - \mathrm{j}b) = 2a$, which is real.

Product: $(a + \mathrm{j}b)(a - \mathrm{j}b) = a^2 - (\mathrm{j}b)^2 = a^2 - (-b^2)$
$= a^2 + b^2$, which is real.

Note the similarity to conjugate surds (section 1). The difference is that with complex numbers, the j^2 causes a change in sign:

$$(a + \mathrm{b})(a - b) = a^2 - b^2$$

$$(a + \mathrm{j}b)(a - \mathrm{j}b) = a^2 + b^2$$

Rule: The complex number $a + \mathrm{j}b$ or $a - \mathrm{j}b$ multiplied by its conjugate is:

$$\boldsymbol{a^2 + b^2}$$

Examples

z	z^*	$z + z^*$	$z\ z^*$
2 + j3	2 – j3	4	13
2 – j5	4 + j5	8	41
–2 + j3	–2 – j3	–4	13
–5 – j7	–5 + j7	–10	74
√2 + j√2	√2 – j√2	2 √2	4

SAQ2-2-5

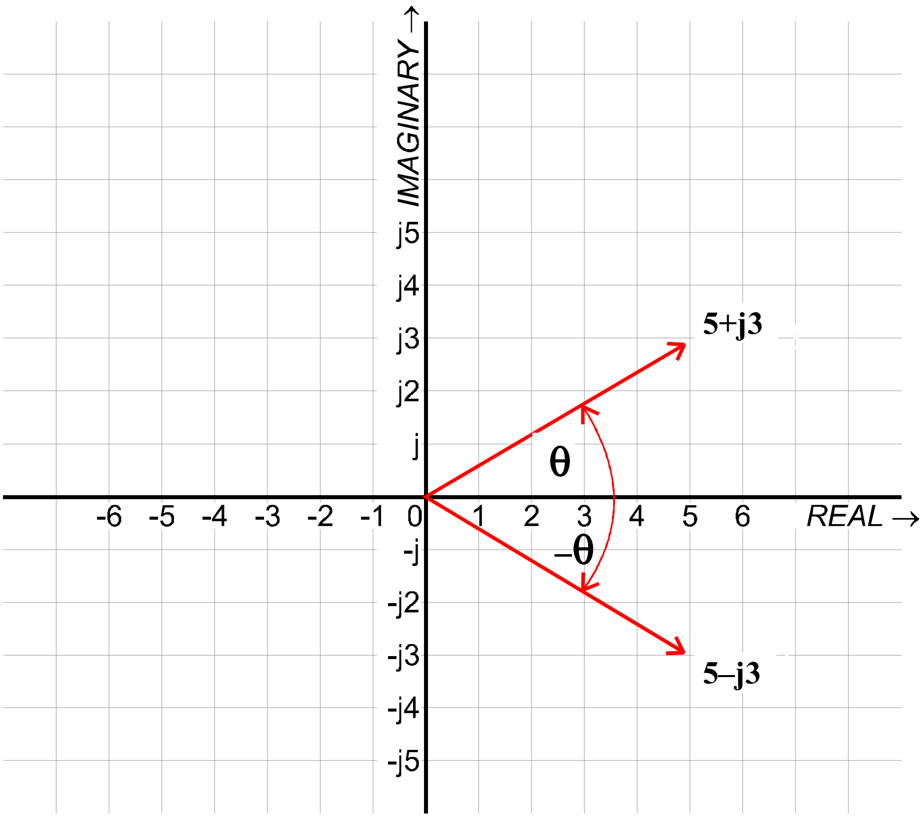
For the values of z in the table, write down

(i) the conjugate, z^* (ii) the sum $z + z^*$ (iii) the product $z\ z^*$

z	z^*	$z + z^*$	$z\ z^*$
$4 + j6$			
$3 - j7$			
$-2 + j5$			
$-9 - j12$			
$2 - j\sqrt{3}$			
$1/\sqrt{2} + j/\sqrt{2}$			
$-\frac{1}{2} - j\,{}^{\sqrt{3}}/_{2}$			

Reflection

The complex conjugate of a number represents a **reflection** in the real axis. Imagine the real axis as a mirror with the vector 5+j3 reflected in it.



p361 fig6

SAQ2-2-6

If z is a complex number, its conjugate z^* represents a reflection in the real axis. What does $-z^*$ represent?

Division of complex numbers

To divide one complex number by another, we turn the divisor into a real number. Therefore we multiply numerator (top) and denominator (bottom) by the complex conjugate of the denominator, ie

$$\frac{x}{y} = \frac{x\,y^*}{y\,y^*}$$

By multiplying numerator and denominator by the same quantity we are, of course, multiplying it by 1, which does not change its value, however, it conveniently turns the denominator into a real number.

Examples

a. $\dfrac{2+j3}{4+j5}$

$$= \frac{(2+j3)(4-j5)}{(4+j5)(4-j5)} = \frac{8-j10+j12+15}{4^2+5^2}$$

$$= \frac{23+j2}{41} = \frac{23}{41} + j\frac{2}{41}$$

$= 0{\cdot}561 + j0{\cdot}049$ to 3 decimal places.

b. $\dfrac{7+j5}{3-j4}$

$$= \frac{(7+j5)(3+j4)}{(3-j4)(3+j4)} = \frac{21+j28+j15-20}{3^2+4^2}$$

$$= \frac{1+j43}{25} = \frac{1}{25} + j\frac{43}{25}$$

$= 0{\cdot}04 + j\,1{\cdot}72$

c. $\dfrac{1}{j} = \dfrac{1\times -j}{j\times -j} = \dfrac{-j}{1} = -j$

This last result is particularly useful, because we can express –j as $\dfrac{1}{j}$

For example, later in AC theory we shall use the expression

$$R - \frac{j}{\omega C} \equiv R + \frac{1}{j\omega C}$$

This formula may be written in either form, whichever is convenient.

SAQ2-2-7

Evaluate the following, expressing the answers in the form $a + \mathrm{j}b$.

a. $$\frac{3+\mathrm{j}8}{1+\mathrm{j}}$$

b. $$\frac{5-\mathrm{j}6}{6-\mathrm{j}8}$$

c. $$\frac{-8-\mathrm{j}7}{-7-\mathrm{j}}$$

d. $$\frac{10}{2+\mathrm{j}}$$

e. $$\frac{1}{\mathrm{R}+\mathrm{j}\omega\mathrm{L}}$$

SAQ2-2-8

The impedance of a series circuit is given by

$$Z = Z_1 + Z_2 + Z_3$$

Calculate Z, given $Z_1 = 1000 + j250$ ohms, $Z_2 = 2200 - j750$ ohms, $Z_3 = 300 - j125$ ohms.

SAQ2-2-9

The impedance of a parallel circuit is given by

$$Z = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

Calculate Z, given $Z_1 = 1{\cdot}0 - j1{\cdot}5$ kΩ, $Z_2 = 5{\cdot}0 + j3{\cdot}2$ kΩ.

SAQ2-2-10

If $\dfrac{1}{Z} = \dfrac{1}{Z_1} + \dfrac{1}{Z_2} + \dfrac{1}{Z_3}$

calculate Z, given $Z_1 = 2 + j3$, $Z_2 = 1 - j$, $Z_3 = 3 + j4$

SAQ2-2-11

Solve the following equation for z.

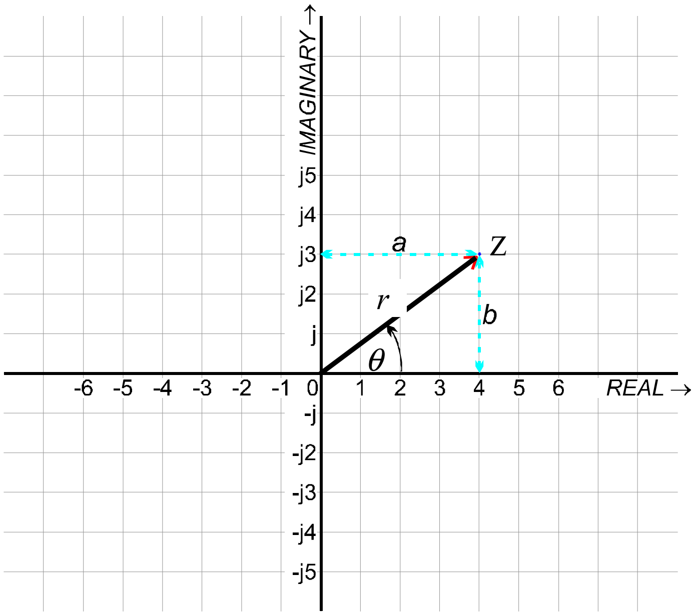
$$\frac{3z}{1-j} \quad + \quad \frac{3z}{j} \qquad \frac{4}{3-j}$$

Chapter 3

Polar form

Polar form of a complex number

The form $a + jb$ of a complex number is called the *rectangular form* or the *Cartesian form*. The number is specified by its Real coordinate a and its Imaginary coordinate b.



p361 fig7

The complex number z could be equally specified by the length from 0 to z, which we shall call r, and the angle θ measured from the positive real axis. (You will recall that a **positive** angle represents an **anticlockwise** rotation.)

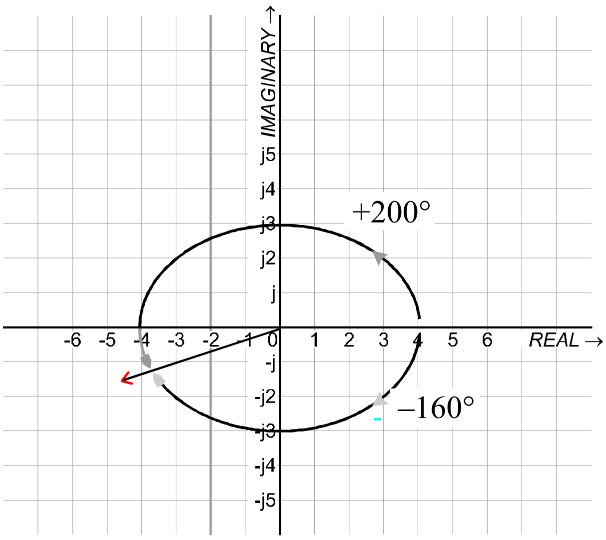
The **polar** form of z is written: $z = r \angle\theta$

modulus

r is called the **modulus** or the **magnitude** of z and written $|z|$. It is a scalar quantity since it measures the length from 0 to z irrespective of the direction. Hence $r \geq 0$.

argument

θ is called the **argument** of z, or simply the **angle**. It is sometimes written as arg(z). It is the angle between the vector from 0 to z and the positive real axis. By convention, a rotation greater than 180º is regarded as a negative (clockwise) angle, so that the numerically smallest value of θ is used. Hence $-180º < \theta \leq 180º$, or in radians $-\pi < \theta \leq \pi$.



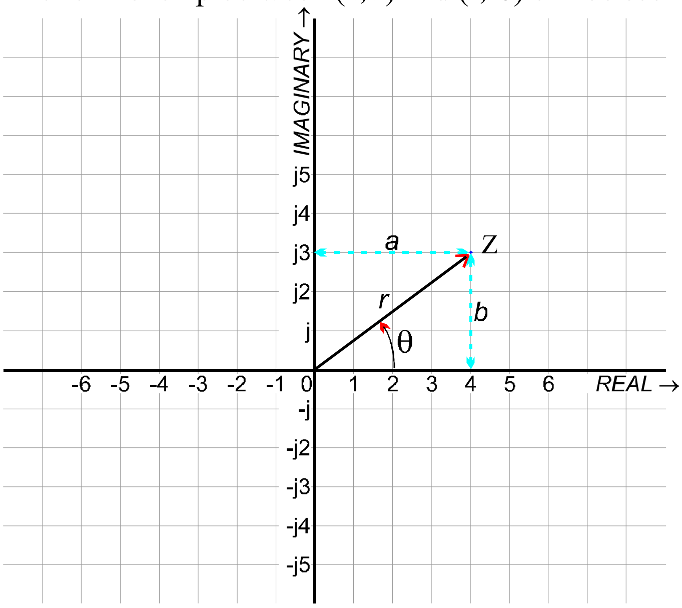
p361 fig8

For example,

$4.5 \angle 200º$ would normally be written as $4.5 \angle -160º$.

Conversion between rectangular and polar forms

The relationship between (a,b) and (r, θ) can be seen from the diagram.



p361 fig7

$a = r\cos\theta$
$b = r\sin\theta$

$$r = \sqrt{a^2 + b^2}$$

$\tan\theta = b/a$

Rules:

To convert from rectangular to polar form

$$r = \sqrt{a^2 + b^2}, \qquad \tan\theta = b/a$$

To convert from polar to rectangular form

$$a = r\cos\theta \qquad b = r\sin\theta$$

Rectangular to polar Examples

a. In the above diagram, $z = 4 + j3$. Convert $4 + j3$ to polar form.

$$r = |z| = \sqrt{4^2 + 3^2} = 5$$

$\tan\theta = 3/4 = 0.75 \;\therefore\; \theta = \tan^{-1} 0.75 = 36{\cdot}9^{\circ}$.

Hence, $4 + j3 = 5 \angle 36{\cdot}9^{\circ}$.

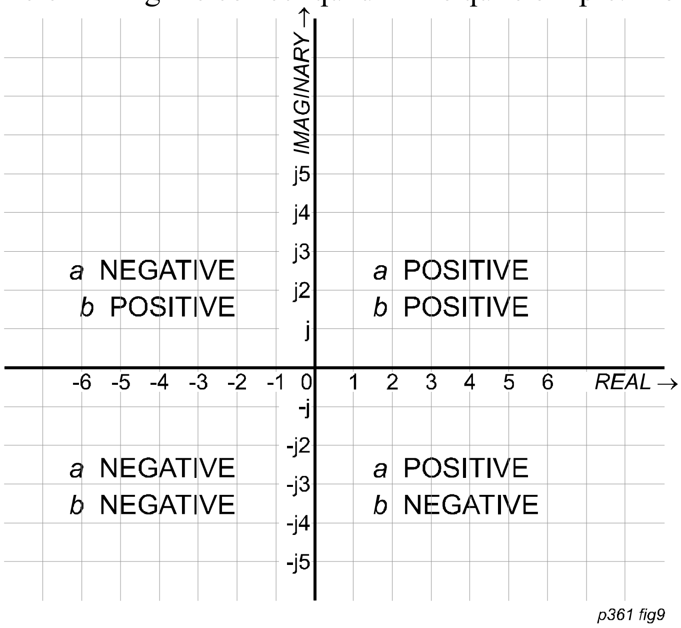
Determining the correct quadrant for the angle

If $\tan\theta = b/a$, it is not necessarily true that $\theta = \tan^{-1}(b/a)$. The function $\tan^{-1} x$ (or arctan x) is defined as having a value between -90° and $+90^{\circ}$.

ie $-90^{\circ} < \tan^{-1} x < 90^{\circ}$
or in radians, $-\pi/2 < \tan^{-1} x < \pi/2$

This is the range of the angle given by the $\tan^{-1}$ function on a calculator. So, for example, although $\tan 135^{\circ} = -1$, if we calculate $\tan^{-1}(-1)$ we get the result $\theta = -45^{\circ}$.

Determining the correct quadrant is quite simple. Looking at the diagram:



If *a, b* are both positive
$0^{\circ} < \theta < 90^{\circ}$

If *a* positive, *b* negative
$-90^{\circ} < \theta < 0^{\circ}$

If *a* negative, *b* positive
$90^{\circ} < \theta < 180^{\circ}$

If *a*, *b* are both negative
$-180^{\circ} < \theta < -90^{\circ}$

Examples

b. Convert $z = -5 + j4$ to polar form.

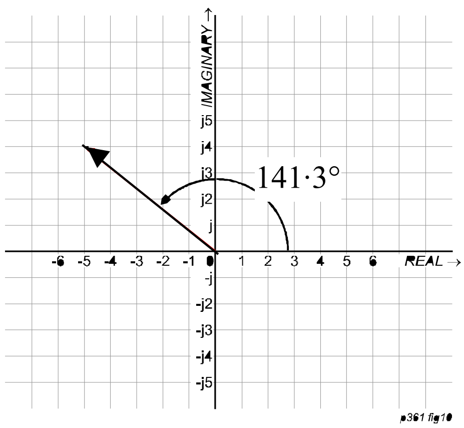
$|z| = = \sqrt{(-5)^2 + 4^2} = 6{\cdot}4$

$\tan\theta = -4/5 = -0{\cdot}8$

$\tan^{-1}(00{\cdot}8) = -38{\cdot}7^{\circ}$ but θ must lie between 90º and 180º.

Hence, $\theta = -38{\cdot}7^{\circ} + 180^{\circ} = 141{\cdot}3^{\circ}$

$\therefore -5 + j4 = 6{\cdot}4 \angle 141{\cdot}3^{\circ}$



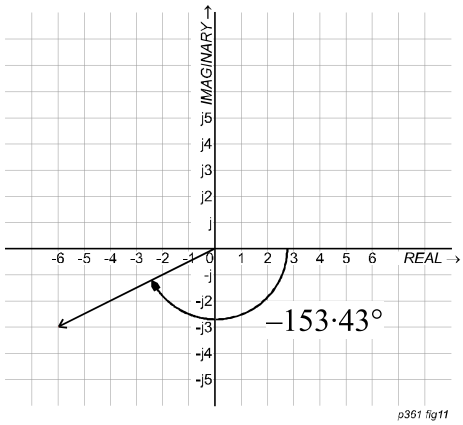
c. Convert $z = -6 - j3$ to polar form

$|z| = = \sqrt{(-6)^2 + (-3)^2} = 6{\cdot}7$

$\tan\theta = -3/(06) = 0{\cdot}5$
$\tan^{-1} 0{\cdot}5 = 26{\cdot}6^{\circ}$ but θ must lie between –90º abd –180º.

Hence, $\theta = 26{\cdot}6^{\circ} - 180^{\circ} = -153{\cdot}4^{\circ}$

$\therefore -6 - j3 = 6{\cdot}7 \angle -153{\cdot}4^{\circ}$



A "mental sketch" will show which quadrant contains the angle.

Example

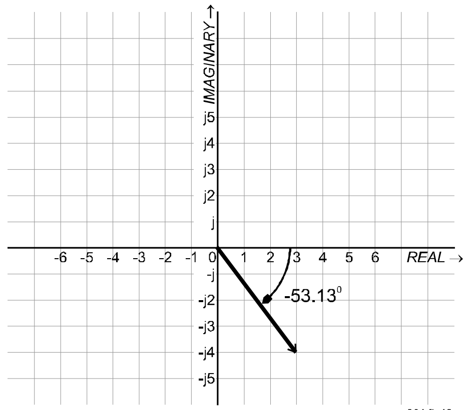
d. Convert $z = 3 - j4$ to polar form

$$|z| = \sqrt{3^2 + (-4)^2} = 5$$

$\tan\theta = -4/3 = -1{\cdot}3333$
$\tan^{-1}(-1{\cdot}3333) = -53{\cdot}1^{o}$

The angle must lie between -90^{o} and 0^{o}
$\therefore -53{\cdot}1^{o}$ is the correct angle.

$\therefore\ 3 - j4 = 5\angle -53{\cdot}1^{o}$



p361 fig12

Rules: If *a* is positive, then $\theta = \tan^{-1}(b/a)$

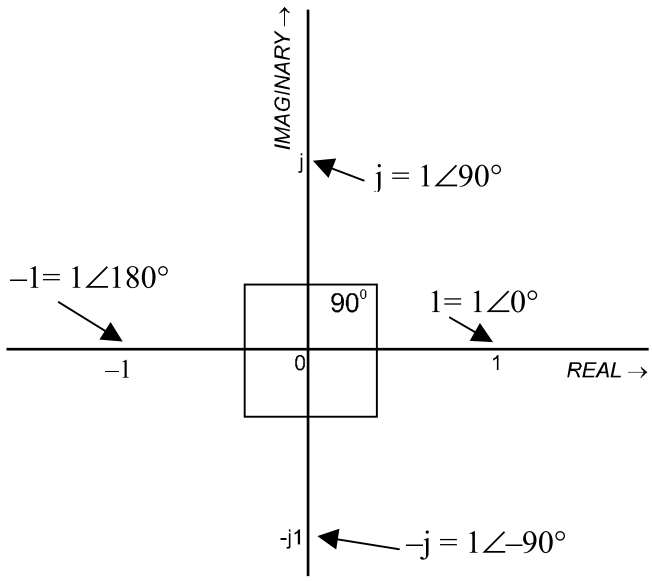
If *a* is negative, then $\theta = \tan^{-1}(b/a) \pm 180^{o}$.
ie add or subtract 180^{o} to $\tan^{-1}(b/a)$, whichever gives the angle in the correct range, $-180^{o} < \theta \le 180^{o}$.

Pure real or pure imaginary complex numbers

What happens if *a* or *b* is zero? The angle is not in any particular quadrant but lies on one of the axes. This happens if the number is purely real or purely imaginary. However it is still a complex number and has a rectangular and a polar form.

For example $z = 0 + j$
We cannot use the formula $\tan^{-1}(1/0)$ since division by zero is not permitted. However, it is clear that:

$$1 = 1\angle 0^{o}, \qquad -1 = 1\angle 180^{o}$$
$$j = 1\angle 90^{o}, \qquad -j = 1\angle -90^{o}$$



p361 fig13

Example

Convert $-j6$ to polar form.
$-j6$ clearly has a magnitude of 6 at an angle of -90^{o}.

$\therefore -j6 = 6\angle -90^{o}$

Polar to rectangular

Polar to rectangular conversion is usually more straightforward since most calculators will evaluate sines and cosines of any sized angle without having to worry about the quadrant.

$$\text{Since,} \quad a = r\cos\theta, \qquad b = r\sin\theta$$

$$r\angle\theta \equiv r\cos\theta + \mathrm{j}\,r\sin\theta$$

$$\equiv r(\cos\theta + \mathrm{j}\sin\theta)$$

Examples

a. Convert $4\angle 30^\circ$ to rectangular form.

$$\begin{aligned} 4\angle 30^\circ &= 4(\cos 30^\circ + \mathrm{j}\sin 30^\circ) \\ &= 4(0{\cdot}866 + \mathrm{j}\,0{\cdot}5) \\ &= 3{\cdot}464 + \mathrm{j}\,2 \end{aligned}$$

b. Convert $5{\cdot}4\angle{-60^\circ}$ to rectangular form.

$$\begin{aligned} 5{\cdot}4\angle{-60^\circ} &= 5{\cdot}4(\cos(-60^\circ) + \mathrm{j}\sin(-60^\circ)) \\ &= 5{\cdot}4(0{\cdot}5 - \mathrm{j}\,0{\cdot}866) \\ &= 2{\cdot}7 - \mathrm{j}\,4{\cdot}677 \end{aligned}$$

c. Convert $6{\cdot}8\angle 135^\circ$ to rectangular form.

$$\begin{aligned} 6{\cdot}8\angle 135^\circ &= 6{\cdot}8(\cos 135^\circ + \mathrm{j}\sin 135^\circ) \\ &= 6{\cdot}8(-0{\cdot}707 + \mathrm{j}\,0{\cdot}707) \\ &= -4{\cdot}808 + \mathrm{j}\,0{\cdot}808 \end{aligned}$$

d. Convert $10\angle{-150^\circ}$ to rectangular form.

$$\begin{aligned} 10\angle{-150^\circ} &= 10(\cos(-150^\circ) + \mathrm{j}\sin(-150^\circ)) \\ &= 10(-0{\cdot}866 - \mathrm{j}\,0{\cdot}5) \\ &= -8{\cdot}66 - \mathrm{j}\,5 \end{aligned}$$

Example

e. Convert $6\angle{-2\pi/3}$ to rectangular form.

It should be remembered that angles are always assumed to be in *radians* unless otherwise specified. eg $\angle 2^{\circ}$ means 2 *degrees*, but $\angle 2$ means 2 *radians*.

$$\begin{aligned} 6\angle{-2\pi/3} &= 6(\cos(-2\pi/3) + j\sin(-2\pi/3)) \\ &= 6(-0{\cdot}5 - j\,0{\cdot}866) \\ &= -3 - j\,5{\cdot}196 \end{aligned}$$

SAQ2-3-1

Convert the following complex numbers to the polar form, $r\angle\theta$, expressing θ in degrees correct to one decimal place.

a. $6 + j8$

b. $-7 + j5$

c. $-2{\cdot}5 - j3{\cdot}6$

d. $5 - j12$

SAQ2-3-2

Express the following complex numbers in polar form:

a. $j2{\cdot}5$

b. $-j7$

c. -5

d. $3{\cdot}8$

SAQ2-3-3

Convert the following complex numbers to polar form, expressing the angle exactly in radians.

a. $3 + j3$

b. $-\sqrt{3} + j$

c. $-2 - j2\sqrt{3}$

SAQ2-3-4

Express in the form $a + \mathrm{j}b$:

a. $5\angle 32^\circ$

b. $6{\cdot}2\angle 140^\circ$

c. $0{\cdot}8\angle -155^\circ$

d. $4{\cdot}9\angle -20^\circ$

e. $3\angle \pi/4$

SAQ2-3-5

Express in rectangular form:

a. $8\angle\pi/3$

b. $5\angle 5\pi/6$

c. $\sqrt{2}\angle -\pi/4$

d. $3\angle -\pi/2$

e. $7{\cdot}52\angle\pi$

Use of Calculators

Note: Most scientific calculators will do polar/rectangular conversion. Before using this facility you should master the methods in this chapter, checking your answers by calculator. Having mastered the theory, you may use the calculator for all subsequent problems and for circuit theory questions. Some calculators will also perform complex arithmetic in rectangular form.

Multiplication and division in polar form

Addition and subtraction of complex numbers must be done in rectangular form, however multiplication and division are much more easily performed in polar form using the following rules:

If $r_1 \angle\theta_1$, $r_2\angle\theta_2$ are 2 complex numbers:

$$r_1\angle\theta_1 \times r_2\angle\theta_1 = r_1r_2\angle(\theta_1 + \theta_2)$$

ie when multiplying; *multiply* the magnitudes and *add* the angles.

$$\frac{r_1\angle\theta_1}{r_2\angle\theta_2} = \frac{r_1}{r_2}\angle(\theta_1 - \theta_2)$$

ie when dividing; *divide* the magnitudes and *subtract* the angles.

Examples

a. $2\angle 20^\circ \times 3\angle 55^\circ = 6\angle 75^\circ$

b. $4\angle -45^\circ \times 5\angle 130^\circ = 20\angle 85^\circ$

c. $1{\cdot}5\angle 80^\circ \times 6\angle 150^\circ = 9\angle 230^\circ = 9\angle(230^\circ - 360^\circ)$
$= 9\angle -130^\circ$

Note: Subtract 360° to make $-180^\circ < \theta \le 180^\circ$

d. $2{\cdot}4\angle -100^\circ \times 3{\cdot}5\angle -150^\circ = 8{\cdot}4\angle -250^\circ = 8{\cdot}4\angle(-250^\circ + 360^\circ)$
$= 8{\cdot}4\angle 110^\circ$

Note: Add 360° to make $-180^\circ < \theta \le 180^\circ$

e. $6\angle 75^\circ \div 3\angle 30^\circ = 2\angle 45^\circ$

f. $7\angle -56^\circ \div 2\angle -150^\circ = 3{\cdot}5\angle 94^\circ$

g. $24\angle 120^\circ \div 6\angle -130^\circ = 4\angle 250^\circ = 4\angle(250^\circ - 360^\circ)$
$= 4\angle -110^\circ$

Note: Subtract 360° to make $-180^\circ < \theta \le 180^\circ$

h. $5{\cdot}5\angle -80^\circ \div 1{\cdot}1\angle 200^\circ = 5\angle -280^\circ = 5\angle(-280^\circ + 360^\circ)$
$= 5\angle 80^\circ$

Note: Add 360° to make $-180^\circ < \theta \le 180^\circ$

Rotating by any multiple of 360° obviously gives the same values of *a* and *b*.

Proof of multiplication and division rules

The proofs are given below of the rules for multiplication and division in polar form. These proofs are given for interest only. You may skip over them if you prefer.

In these proofs, we make use of the trigonometric identities:

$\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$

$\cos^2 A + \sin^2 A \equiv 1$

Multiplication

$r_1\angle\theta_1 \times r_2\angle\theta_2 = r_1(\cos\theta_1 + j\sin\theta_1)\ r_2(\cos\theta_2 + j\sin\theta_2)$

$= r_1 r_2(\cos\theta_1 + j\sin\theta_1)(\cos\theta_2 + j\sin\theta_2)$

$= r_1 r_2\ \{\cos\theta_1\cos\theta_2 + j^2\sin\theta_1\sin\theta_2 + j\sin\theta_1\cos\theta_2 + j\cos\theta_1\sin\theta_2\}$

$= r_1 r_2\ \{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + j(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\}$

$= r_1 r_2\ \{\cos\theta_1 + \cos\theta_2) + j\sin(\theta_1 + \theta_2)\}$

$= r_1 r_2\angle(\theta_1 + \theta_2)$

Division

$$\frac{r_1\angle\theta_1}{r_2\angle\theta_2} = \frac{r_1}{r_2}\frac{(\cos\theta_1 + j\sin\theta_1)}{(\cos\theta_2 + j\sin\theta_2)}$$ (multiply top and bottom by conjugate)

$$= \frac{r_1}{r_2}\ \frac{(\cos\theta_1 + j\sin\theta_1)(\cos\theta_2 - j\sin\theta_2)}{(\cos\theta_2 + j\sin\theta_2)(\cos\theta_2 - j\sin\theta_2)}$$

$$= \frac{r_1}{r_2}\ \frac{\cos\theta_1\cos\theta_2 - j^2\sin\theta_1\sin\theta_2 + j\sin\theta_1\cos\theta_2 - j\cos\theta_1\sin\theta_2}{\cos^2\theta_2 - j^2\sin^2\theta_2}$$

$$= \frac{r_1}{r_2}\ \frac{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + j(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$= \frac{r_1}{r_2}\ \frac{\cos(\theta_1 - \theta_2) + j\sin(\theta_1 - \theta_2)}{1}$$

$$= \frac{r_1}{r_2}\angle(\theta_1 - \theta_2)$$

SAQ2-3-6

Evaluate in polar form:

a. $3{\cdot}2\angle 80^\circ \times 4{\cdot}5\angle 23^\circ$

b. $7{\cdot}4\angle 120^\circ \times 8\angle 75^\circ$

c. $8{\cdot}2\angle -\pi/6 \times 3{\cdot}5\angle 2\pi/3$

d. $9{\cdot}5\angle -40^\circ \times 3\angle -175^\circ$

e. $2{\cdot}2\angle \pi \times 7{\cdot}4\angle \pi/4$

f. $4{\cdot}8\angle 135^\circ \div 3{\cdot}2\angle 70^\circ$

g. $3{\cdot}28\angle 150^\circ \div 16{\cdot}4\angle -80^\circ$

h. $19\angle -100^\circ \div 2\angle 80^\circ$

i. $15\angle 3\pi/4 \div 4\angle -2\pi/3$

Chapter 4

Exponential form and De Moivre's theorem

De Moivre's theorem

It follows from the rule for multiplication that:

$$(r\angle\theta)^2 = r \times r\,\angle(\theta+\theta) = r^2\angle 2\theta$$

$$(r\angle\theta)^3 = r\angle\theta \times (r\angle\theta)^2 = r^3\angle 3\theta$$

We can see that by successive multiplication that $(r\angle\theta)^n = r^n\angle n\theta$

If $r = 1$, we can write the above as $(\cos\theta + \mathrm{j}\sin\theta)^n = (\cos n\,\theta + \mathrm{j}\sin n\,\theta$

This is called de Moivre's theorem. It can be shown that it is true for any value of n, not just positive integers. De Moivre's theorem and the rules for multiplication and division in polar form, can be proved more directly from the exponential form of a complex number which we shall now consider.

Exponential form of a complex number

Something about the above rules may seem familiar from Section 1, chapter 3. When multiplying we *add* the angles. When dividing we *subtract* the angles. When raising to a power we *multiply* the angle by the power. These look like the rules of **indices**. This is no coincidence, since θ is in fact an imaginary index. The exponential form, sometimes called Euler's identity is:

$$\mathbf{\cos\theta + j\sin\theta \equiv e^{j\theta}}$$

in the form $e^{j\theta}$, θ is always measured in **radians.**

Thus the exponential form of a complex number is

$$r\angle\theta \equiv r\mathrm{e}^{\mathrm{j}\theta}$$

θ measured in **radians**

A proof of Euler's identity is given on the next page, however, this identity is often taken as a definition of $e^{j\theta}$. All the trigonometric identities can be derived from it.

The proof is given for interest only and you may skip it if you wish. The exponential form is very important in signal processing theory and should be committed to memory.

Proof of Euler's identity

Let $z = \cos\theta + j\sin\theta$

Differentiating with respect to θ ;

$$\frac{dz}{d\theta} = -\sin\theta + j\cos\theta$$

$$= j^2\sin\theta + j\cos\theta$$

$$= j(\cos\theta + j\sin\theta)$$

$$= jz$$

$$\therefore \quad \frac{dz}{d\theta} = jz$$

Integrating, $\int \frac{dz}{z} = \int j\, d\theta$

$\ln z = j\theta + c$ where c is an arbitrary constant.

Hence, $z = e^{j\theta + c}$

$\therefore \quad \cos\theta + j\sin\theta = e^{j\theta + c}$

To determine c, put $\theta = 0$, giving $1 = e^c \;\therefore\; c = 0$

Hence, $\cos\theta + j\sin\theta = e^{j\theta}$

The rules for multiplication, division, and De Moivre's theorem now follow directly from the rules of indices, ie

Multiplication

$$r_1\angle\theta_1 \times r_2\angle\theta_2 = r_1e^{j\theta_1} \times r_2e^{j\theta_2} = r_1r^2\, e^{j(\theta_1+\theta_2)}$$

$$= r_1r_2\angle(\theta_1+\theta_2)$$

Division

$$r_1\angle\theta_1 \div r_2\angle\theta_2 = r_1e^{j\theta_1} \div r_2e^{j\theta_2} = \frac{r_1}{r_2}\, e^{j(\theta_1-\theta_2)}$$

$$= \frac{r_1}{r_2}\angle(\theta_1-\theta_2)$$

De Moivre

$$(r\angle\theta)^n = (r\,e^{j\theta})^n = r^n\,e^{jn\theta} = r^n\angle n\theta \qquad \text{(for any } n\text{)}$$

SAQ2-4-1

Write the following complex numbers in the form $r\,e^{j\theta}$.

a. $4{\cdot}5\angle 30^{\circ}$

b. $2{\cdot}5 - j1{\cdot}2$

c. $-10 - j12$

d. $2 + j2\sqrt{3}$

Powers of complex numbers

De Moivre's theorem may be used to find powers of complex numbers which would be very laborious in rectangular form.

Example

Evaluate $(0{\cdot}9 + j1{\cdot}2)^7$

Expanding this in rectangular form would take some time.
In polar form, $0{\cdot}9 + j1{\cdot}2 = 1{\cdot}5\angle 53{\cdot}13^{o}$.

By De Moivre's theorem, $(1{\cdot}5\angle 53{\cdot}13^{o})^7 = 1{\cdot}5^7\angle 53{\cdot}13^{o} \times 7$
$= 17{\cdot}09\angle 372^{o} = 17{\cdot}09\angle 12^{o}$

$= 16{\cdot}72 + j3{\cdot}53$

Complex conjugate

$1/e^{j\theta} = e^{-j\theta}$; ie $e^{j\theta}$ and $e^{-j\theta}$ are inverses of each other.

$e^{j\theta}$ and $e^{-j\theta}$ are also complex conjugates of each other.

Proof: From Euler's identity, $e^{j\theta} \equiv \cos\theta + j\sin\theta$

You should recall from trigonometry that
$\cos(-\theta) = \cos\theta$, ie cosine is an *even* function.
$\sin(-\theta) = -\sin\theta$, ie sine is an *odd* function.

Hence, $e^{-j\theta} \equiv \cos\theta - j\sin\theta$, which is the conjugate of $e^{j\theta}$.

SAQ2-4-2

Using De Moivre's theorem evaluate the following in polar form and convert to rectangular form.

a. $(2\angle 20^{o})^3$

b. $(3\angle -100^{o})^4$

c. $(2 + j3)^6$

d. $(-3 - j4)^5$

e. $(3\angle -\pi/3)^2$

Exponential form of sine and cosine

Above, we proved the important identity

$$\cos\theta + j\sin\theta \equiv e^{j\theta}$$

Remember that θ is measured in **radians.**

Putting θ equal to *x* and = –*x* in the identity, we get:

$\cos x + j\sin x \equiv e^{jx}$ .... (1).

$\cos x - j\sin x \equiv e^{-jx}$ .... (2).

Adding (1) and (2) we obtain $2\cos x \equiv e^{jx} + e^{-jx}$

Hence, $\cos x \equiv \dfrac{e^{jx} + e^{-jx}}{2}$

Subtracting (2) from (1) we obtain $2j\sin x \equiv e^{jx} - e^{-jx}$

Hence, $\sin x \equiv \dfrac{e^{jx} + e^{-jx}}{2j}$

These two expressions may be taken as definitions of the circular functions, sine and cosine. They are very important in signal processing theory and should be remembered. To emphasise their importance, they are repeated below. *x* is of course measured in radians.

$$\cos x \equiv \frac{e^{jx} + e^{-jx}}{2}$$

$$\sin x \equiv \frac{e^{jx} - e^{-jx}}{2j}$$

As 1/j = –j, we can also write the expression for sin *x* as:

$$\sin x \equiv j\tfrac{1}{2}(e^{-jx} - e^{jx})$$

It should be appreciated that although sin x and cos x are defined in terms of complex numbers, that the sines and cosines of real numbers are real. Why is this so? You will recall from chapter 1 that the sum of a complex number and its conjugate is purely real. Also the difference of a complex number and its conjugate is purely imaginary.

We have seen that e^{jx} and e^{-jx} are complex conjugates.

$\therefore\ e^{jx} + e^{-jx}$ must be real, hence $\frac{1}{2}(e^{jx} + e^{-jx})$ is real.

$e^{jx} - e^{-jx}$ must be imaginary, hence $\dfrac{1}{2j}(e^{jx} - e^{-jx})$ is real.

SAQ2-4-3

Given that $\tan x \equiv \dfrac{\sin x}{\cos x}$

Write down expressions for tan x in terms of

a. e^{jx} and e^{-jx}

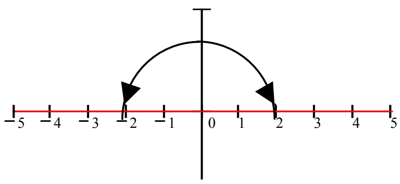
b. e^{j2x} and e^{-j2x}

Chapter 5

Roots of complex numbers

Roots of a complex number

We have seen that every real number has 2 square roots. For example, the square roots of 4 are $\pm\sqrt{4} = \pm 2$.



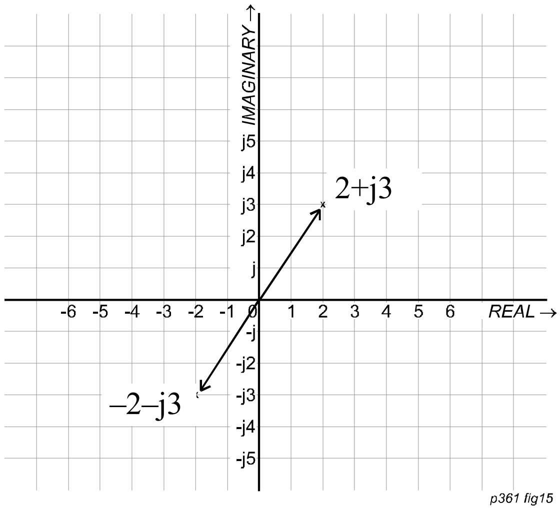
These roots are180º apart, since multiplication by –1 represents a rotation of 180º (c.f. Section 1, chapter 1).

Similarly, every complex number (which includes the real numbers) has 2 square roots.

Consider the complex number –5 + j12. This has the square roots 2 + j3 and –2 –j3 . Check:

$(2 + j3)^2 = 2^2 + (j3)^2 + 2 \times 2 \times j3 = 4 - 9\ j12 \qquad = \qquad -5 + j12$

$(-2 - j3)^2 = (-2)^2 + (-j3)^2 + 2 \times (-2) \times (-j3) = 4 - 9 + j12 \qquad = \qquad -5 + j12$



Note that these roots also are 180º apart, since each root is –1 times the other.
ie the square roots of –5 + j12 are ±(2 + j3).

The 2 roots have the same modulus.
$2 + j3 = 3{\cdot}6\angle 56{\cdot}3^\circ$
$-2 - j3 = 3{\cdot}6\angle(56{\cdot}3^\circ - 180^\circ)$
$= 3{\cdot}6\angle -123{\cdot}7^\circ$

> The 2 square roots of any complex number have the same modulus and their angles are 180º apart.

This can be proved from De Moivre's theorem.

Consider the 2 numbers, $z_1 = r\angle\theta$, $z_2 = r\angle(\theta \pm 180^\circ)$, which have the same modulus, r, and are separated by 180º.

$z_1^2 = r^2\angle 2\theta, \quad z_2^2 = r^2\angle(2\theta \pm 360^\circ)$ by De Moivre.

Now, $\cos(\phi \pm 360^\circ) + j\sin(\phi \pm 360^\circ) \equiv \cos\phi + j\sin\phi$
Hence, $z_1^2 = z_2^2$. $\therefore$ z_1 and z_2 are both square roots of the same number.

Furthermore, since they have the same modulus, r, and there is a rotation of 180º between them; $z_2 = -z_1$.

Finding the square root of a complex number

De Moivre's theorem gives us a way of finding square roots. Putting $n = ½$,

$$(r\angle\theta)^{1/2} = r^{1/2}\angle ½\theta$$

Hence, $r^{1/2}\angle ½\theta$ is a square root of $r\angle\theta$. This is called the principal value. The other root is the negative of this, which is $r^{1/2}\angle(½\theta \pm 180º)$. Whether we add or subtract 180º depends which gives us an angle in the conventional range of $-18 < \theta \leq 180º$.

Examples

a. Find the square roots of $9\angle 60º$

The principal root is $\sqrt{9}\angle ½\times 60º = 3\angle 30º = 2{\cdot}6 + j1{\cdot}5$

The other root is $3\angle(30º - 180º) = 3\angle -150º = -2{\cdot}6 - j1{\cdot}5$

In this instance, we *subtract* 180º giving –150º, rather than adding which would give 210º.

Hence the square roots are $\pm(2{\cdot}6 + j1{\cdot}5)$.

b. Find the square roots of $-3 + j4$

Converting to polar form, $-3 + j4 = 5\angle 126{\cdot}87º$

The principal root is $\sqrt{5}\angle ½\times 126{\cdot}87º = 2{\cdot}236\angle 63{\cdot}43º = 1 + j2$

The other root is $2{\cdot}236\angle(63{\cdot}43º - 180º) = 2{\cdot}236\angle -116{\cdot}57º = -1 - j2$

Hence the square roots of $-3 + j4$ are $\pm(1 + j2)$

c. Find the square roots of $-12 - j35$

Converting to polar form, $-12 - j35 = 37\angle -108{\cdot}92º$

The principal root is $\sqrt{37}\angle ½\times -108{\cdot}92º = 6{\cdot}08\angle -54{\cdot}46º = 3{\cdot}54 - j4{\cdot}95$

The other root is $6{\cdot}08\angle(-54{\cdot}46º + 180º) = 6{\cdot}08\angle 125{\cdot}54º = 3{\cdot}54 + j4{\cdot}95$

In this instance, we *add* 180º to give 125·54º

Hence the square roots of $-12 - j35$ are $\pm(3{\cdot}54 - j4{\cdot}95)$.

It should be evident, by now, that we only need to find the principal value in rectangular form and multiply it by –1 to give the other root.

SAQ2-5-1

Find the 2 square roots of the following numbers and express the answers in rectangular form.

a. $25\angle -120^{\circ}$

b. $5 - j12$

c. $-24 - j70$

d. $6 + j8$

e. $-j9$

Further roots of complex numbers

Cube Roots

A complex number has 3 cube roots. They have the same modulus and are separated by 360º ÷ 3 = 120º. Again, this can be proved by De Moivre's theorem.

$z_1 = r\angle\theta$, $z_2 = r\angle(\theta + 120^\circ)$, $z_3 = r\angle(\theta - 120^\circ)$

are 3 complex numbers of the same modulus, r, separated by 120º. By De Moivre's theorem:

$z_1^3 = r^3\angle 30$

$z_2^3 = r^3\angle(30+360^\circ)$

$z_3^3 = r^3\angle(30-360^\circ)$

Now, $\cos(\phi\pm360^\circ) + j\sin(\phi\pm360^\circ) \equiv \cos\phi + j\sin\phi$

hence, $z_1^3 = z_2^3 = z_3^3$

$\therefore$ z_1, z_2, z_3 are all cube roots of the same number.

Finding the cube root of a complex number

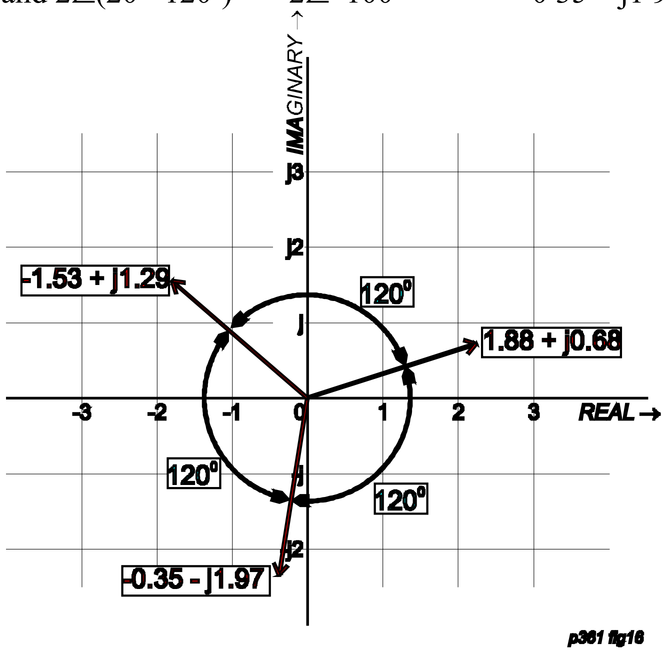
Therefor, by De Moivre's theorem one cube root of $r\angle\theta$ is $r^{1/3}\angle\theta\div3$.
The other 2 roots are $r^{1/3}\angle(\theta\div3 + 120^\circ)$ and $r^{1/3}\angle(\theta\div3 - 120^\circ)$.

Example

a. Find the cube roots of $8\angle60^\circ$

One cube root is $8^{1/3}\angle60^\circ\div3 = 2\angle20^\circ = 1{\cdot}88 + j\,0{\cdot}68$

The other roots are $2\angle(20^\circ+120^\circ) = 2\angle140^\circ = 1{\cdot}53 + j1{\cdot}29$
and $2\angle(20^\circ-120^\circ) = 2\angle-100^\circ = -0{\cdot}35 - j1{\cdot}97$



The 3 cube roots are shown on the Argand diagram, each of magnitude 2, separated by angles of 120º.

nth root of a complex number

It should now be clear that the complex number $r\angle\theta$ will have *n nth* roots each of modulus $r^{1/n}$ separated by angles of 360º÷*n*. On the Argand diagram the roots will lie on a circle of radius $r^{1/n}$, spaced equally around the circle at angular intervals of 360º÷*n*.

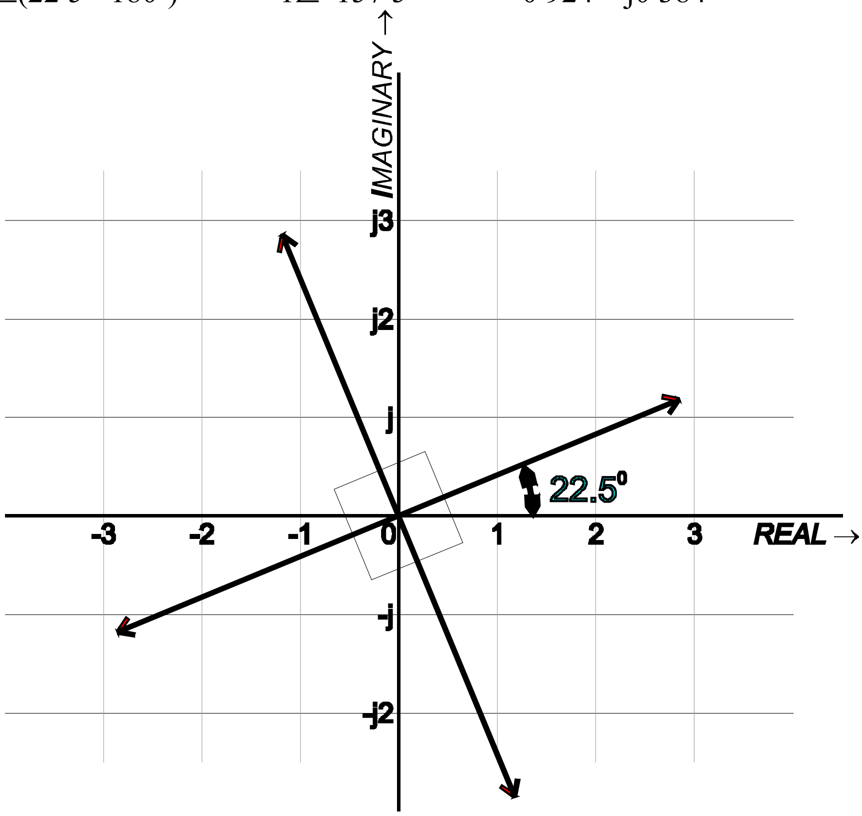
Find the four 4th roots of j.

Example

Converting to polar form, j = 1∠90º

The principal 4th root is 1¼∠90º÷4 = 1∠22·5º = 0·924 + j0·383
The other 3 roots are found by adding or subtracting multiples of 360÷4 = 90º.
The other roots are:

1∠(22·5º+90º)	=	1∠112·5º	=	–0·383 + j0·924
1∠(22·5º–90º)	=	1∠–67·5º	=	0·383 – j0·924
1∠(22·5º–180º)	=	1∠–157·5º	=	–0·924 – j0·384



p361 fig17

The roots are shown on the Argand diagram, lying on a circle of unit radius, spaced apart by 90º.

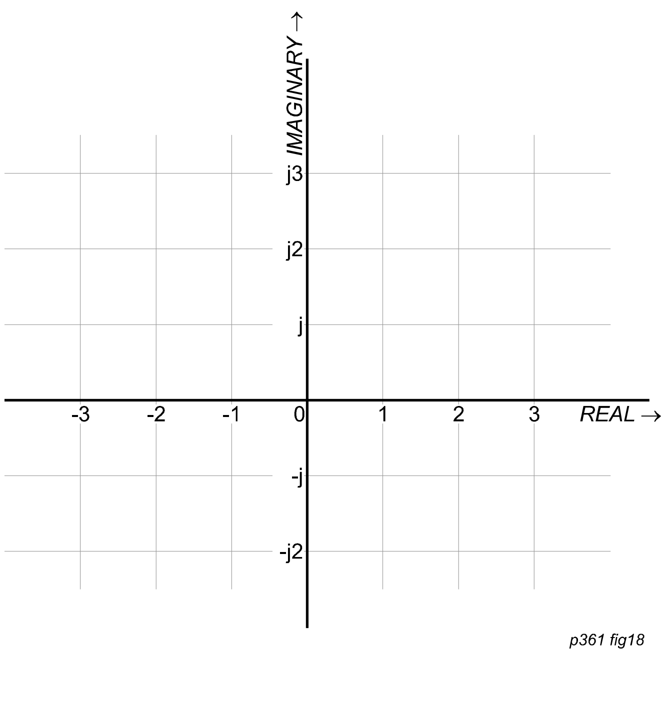
You may have spotted that the roots may be found by multiplying the rectangular form successively by j. This is, of course, because multiplication by j rotates by 90º.

SAQ2-5-2 Find the 3 cube roots of the following complex numbers and express the results in rectangular form.

a. $125\angle{-150^\circ}$

b. $-610 - j182$

SAQ2-5-3 Find the 3 cube roots of –1 in rectangular form and sketch them on the Argand diagram.



p361 fig18

SAQ2-5-4

The characteristic impedance of a transmission line is given by:

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

Evaluate the principal value of Z_0 where
R = 5 ohms, G = 2 × 20^{-6} siemens, L = 10^{-5} henrys, C = 3 × 10^{-12} farads,
$\omega = 2\pi \times 10^6$ rad/s.

SAQ2-5-5

The propagation coefficient of a transmission line is defined as

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

Evaluate the principal value of γ where R = 50, L = 0·0004, C = 2 × 10^{-12},
G is negligible, $\omega = 2\pi \times 16000$

Chapter 6

Equating parts

Equating real and imaginary parts

In chapter 1 we saw that the real and imaginary numbers coincide only at zero. A real number has no imaginary part and an imaginary number has no real part. This enables us to equate the real and imaginary parts of complex numbers.

$$\text{If} \quad a + \text{j}b = c + \text{j}d$$
$$\text{then} \quad a = c \quad \text{and} \quad b = d$$

ie 2 complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal.

It follows that if $a + \text{j}b = 0$, then $a = 0$ and $b = 0$.

Thus an equation in a complex variable is actually 2 equations in one. This process has particular applications in circuit theory where we have 2 quantities which are in quadrature and we can solve for both at once.

Example

Find a and b in the equation

$$\frac{a + 2}{2a + \text{j}b} = 1 - \text{j}3$$

Multiplying both sides by $2a + \text{j}b$;

$$a + 2 = (2a + \text{j}b)(1 - \text{j}3)$$
$$a + 2 = 2a + 3b - 6\,\text{j}a + \text{j}b$$
$$2 = a + 3b + \text{j}(-6a + b)$$

Hence, $a + 3b = 2$ and $-6a + b = 0$
Solving this pair of simultaneous equations gives $a = {}^{2}/_{19}$, $b = {}^{12}/_{19}$.

Example

The condition for balance of a 4 arm bridge is:

$$\frac{Z_1}{Z_2} = \frac{Z_3}{Z_4}$$

It is used to measure the unknown inductance L_x and Resistance R_x of a coil, in terms of known components.

$Z_1 = R_x + j\omega L_x$ ohms, is the impedance of the unknown coil.

$Z_2 = 2 + j\omega\, 0{\cdot}1$ ohms, is the impedance of a standard coil.

$Z_3 = 94{\cdot}5\ \Omega$ is a known resistance.

$Z_4 = 25\ \Omega$ is a known resistance.

We can therefore write the equation

$$\frac{R_x + j\omega L_x}{2 + j\omega\, 0\cdot 1} = \frac{94\cdot 5}{25}$$

$$R_x + j\omega L_x = 3{\cdot}78(2 + j\omega\, 0{\cdot}1)$$

$$R_x + j\omega L_x = 7{\cdot}56 + j\omega\, 0{\cdot}378$$

Equating real parts: $R_x = 7{\cdot}56$ ohms

Equating imaginary parts: $\omega L_x = \omega\, 0{\cdot}378$

$\therefore L_x = 0{\cdot}378$ Henrys.

You may note that this measurement is independent of the frequency ω at which it is performed.

This technique of equating real and imaginary parts enables us to solve for 2 unknowns which are in quadrature, at the same time.

SAQ2-6-1

If $(a + \mathrm{j}b)^2 + b^2 = 4 + \mathrm{j}12$ where a is positive, find a and b.

SAQ2-6-2

The condition for balance of a 4 arm bridge is

$$\frac{Z_1}{Z_2} = \frac{Z_3}{Z_4}$$

If $Z_1 = R_x - \mathrm{j}/(\omega C_x)$

$Z_2 = 0{\cdot}1 - \mathrm{j}/(\omega\, 3{\cdot}5 \times 10^{-6})$

$Z_3 = 24$

$Z_4 = 50$

Find the values of R_x and C_x

Chapter 7

Complex roots of equations

Quadratic equations

Section 1, chapter 4 discussed the real roots of quadratic equations. You will recall that a quadratic equation is of the form

$$ax^2 + bx + c = 0$$

and has 2 roots which are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

You will also recall that the expression $b^2 - 4ac$ is called the **discriminant** and that if the discriminant is negative it has no real square root. Thus, even if the coefficients *a, b* and *c* are real, the equation has no real solution.

However, we know that the square root of a negative real number may be expressed as an "imaginary" number, and so such an equation has a complex solution.

Example

Solve the equation $x^2 - 4x + 13 = 0$

Applying the formula:

$$x = \frac{4 \pm \sqrt{(4^2 - 4 \times 1 \times 13)}}{2 \times 1}$$

$$= \frac{4 \pm \sqrt{(16 - 52)}}{2}$$

$$= \frac{4 \pm \sqrt{-36}}{2}$$

$$= \frac{4 \pm j6}{2} = 2 \pm j3$$

Thus the 2 roots are $2 + j3$ and $2 - j3$. It is evident that if the roots are complex, then they will be *complex conjugates.*

We can state as a rule:

The equation $ax^2 + bx + c = 0$
where *a, b, c* are real numbers,
has complex conjugate roots if $b^2 - 4ac < 0$

SAQ2-7-1

Solve the quadratic equation

$$2x^2 + 12x + 50 = 0$$

SAQ2-7-2

Solve the following quadratic equation, expressing the roots to 2 decimal places.

$$3x^2 - 4x + 2 = 0$$

Complex factors

In Section 1, chapter 3, we also saw that a quadratic expression may be resolved into 2 linear factors. This is restated below.

$$ax^2 + bx + c \equiv a(x - \alpha)(x - \beta)$$

where α, β are the roots of the quadratic equation $ax^2 + bx + c = 0$

If the roots, α, β are complex, then the factors are complex.

Example

Factorize $x^2 + 4x + 13$

$x^2 + 4x + 13 \equiv (x - \alpha)(x - \beta)$

where α, β are the roots of $x^2 + 4x + 13 = 0$

$$\text{Hence, } \alpha, \beta = \frac{-4 \pm \sqrt{(16 - 52)}}{2} = \frac{-4 \pm \sqrt{-36}}{2}$$

$$= \frac{-4 \pm j6}{2} = -2 \pm j3$$

Therefore the factors are $\{x - (-2 + j3)\}\{x - (-2 - j3)\}$

$= (x + 2 - j3)(x + 2 + j3)$

Example

Factorize $4x^2 - 4x + 5$

$4x^2 - 4x + 5 \equiv 4(x - \alpha)(x - \beta)$

where α, β are the roots of $4x^2 - 4x + 5 = 0$

$$\text{Hence, } \alpha, \beta = \frac{4 \pm \sqrt{(16 - 80)}}{8} = \frac{4 \pm \sqrt{-64}}{8}$$

$$= \frac{4 \pm j8}{8} = \tfrac{1}{2} \pm j$$

Hence factors are $4(x - \tfrac{1}{2} - j)(x - \tfrac{1}{2} + j)$

$= 2(x - \tfrac{1}{2} - j)\ 2(x - \tfrac{1}{2} + j)$

$= (2x - 1 - j2)(2x - 1 + j2)$

SAQ2-7-3

Resolve into complex factors

a. $x^2 - 10x + 26$

b. $9x^2 - 12x + 13$

c. $2x^2 + 8$

Factors of higher polynomials

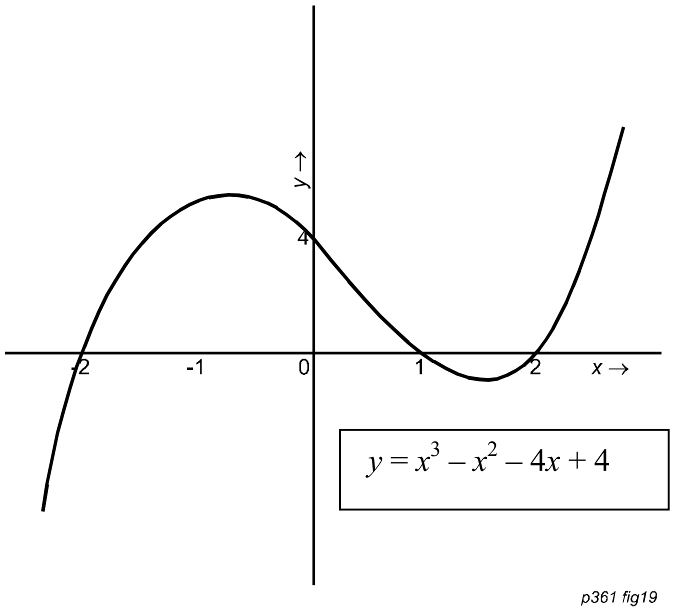
The above principles may be extended to polynomials of higher degree. For example, consider a third degree (cubic) polynomial.

$$ax^3 + bx^2 + cx + d \;\equiv\; a(x-\alpha)(x-\beta)(x-\gamma)$$

where α, β, γ are the roots of $ax^3 + bx^2 + cx + d = 0$

Cubic functions

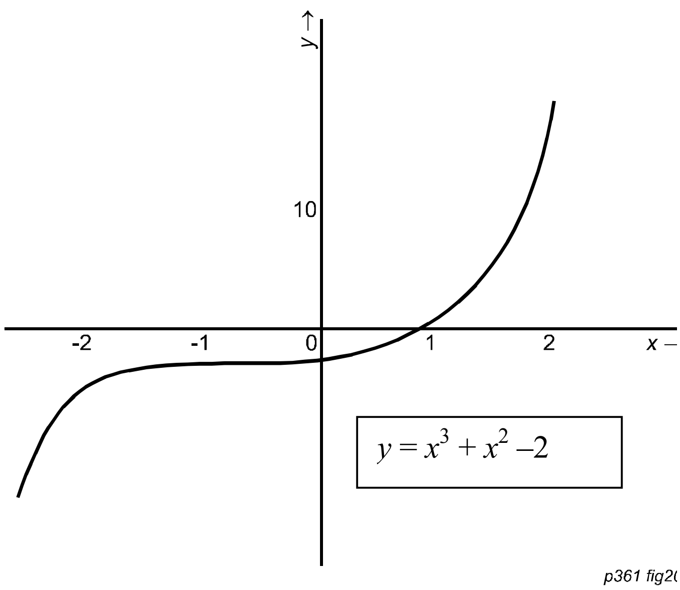
A cubic equation always has at least one real root. The other 2 are either both real (unequal or equal) or are both complex (conjugate).



This cubic function has 3 real factors:

$(x+2)(x-1)(x-2)$

Hence, y = 0 at $x = -2, 1, 2$



This cubic function has one real factor and 2 complex:

$(x-1)(x+1-\mathrm{j})(x+1+\mathrm{j})$

Hence, y = 0 at $x = 1$ only.

Similarly, a fourth degree polynomial has 4 linear factors which may be all real, all complex, or 2 real and 2 complex. The complex roots always occur in conjugate pairs.

Polynomials of degree n

In general, a polynomial of the n^{th} degree has n linear factors, ie
If P_n is a polynomial of degree n with real coefficients.

$$P_n = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots\cdots + a_nx^n$$

$$\equiv a_n\underbrace{(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)(x-\varepsilon)\cdots\cdots(x}_{n \text{ factors}}-\zeta)$$

where $\alpha, \beta, \gamma, \delta, \varepsilon, + \cdots\cdots, \zeta$, are the n roots of the equation $P_n = 0$ which may be real or complex. Complex roots always occur in conjugate pairs. If n is odd, then at least one root is real. If any of the roots are equal then the corresponding factor is repeated. Such a root is called a repeated root. Repeated roots are always real. The graph will touch the x axis without crossing, at a repeated root.

There are occasions in the study of networks where we may wish to resolve a polynomial into real and/or imaginary factors. The solution of polynomial equations higher than quadratics is very difficult, and equations of degree higher than 4 can usually only be solved by numerical methods on a computer. Such methods will be used later on your course.

If some of the roots are known, it may be possible to extract the others by division.

Example

The third degree polynomial $x^3 - 8x^2 + 37x - 50$ has one real factor $(x - 2)$ and 2 complex factors. Find all the factors.

Applying algebraic division (refer Section 1 chapter 4.

$$\begin{array}{r|l} & x^2 - 6x + 25 \\ \hline x - 2 & x^3 - 8x^2 + 37x - 50 \\ & \underline{x^3 - 2x^2} \\ & -6x^2 + 37x - 50 \\ & \underline{-6x^2 + 12x} \\ & 25x - 50 \\ & \underline{25x - 50} \\ & \underline{\underline{0}} \end{array}$$

There is no remainder, hence $x^3 - 8x^2 + 37x - 50 \equiv (x - 2)(x^2 - 6x + 25)$

Now $x^2 - 6x + 25 \equiv (x - \alpha)(x - \beta)$
where α, β are the roots of $x^2 - 6x + 25 = 0$. $\therefore \alpha, \beta$ $= \dfrac{6 \pm \sqrt{(36 - 100)}}{2}$

$= 3 \pm j4$

Hence, $x^3 - 8x^2 + 37x - 50 \equiv (x - 2)(x - 3 - j4)(x - 3 + j4)$

Example

The equation $x^4 + 8x^3 + 23x^2 + 30x + 18 = 0$ has a repeated root at $x = -3$ and 2 complex roots. By division and solving the quadratic equation, find all the roots.

If -3 is a repeated root, then $(x + 3)(x + 3)$ must be factors.

Dividing the polynomial by $(x + 3)^2$,

$$
\begin{array}{r l}
 & \quad x^2 + 2x + 2 \\
x^2 + 6x + 9 \,) & x^4 + 8x^3 + 23x^2 + 30x + 18 \\
 & \underline{x^4 + 6x^3 + 9x^2} \\
 & \qquad 2x^3 + 14x^2 + 30x + 18 \\
 & \qquad \underline{2x^3 + 12x^2 + 18x} \\
 & \qquad\qquad 2x^2 + 12x + 18 \\
 & \qquad\qquad \underline{2x^2 + 12x + 18} \\
 & \qquad\qquad\qquad 0
\end{array}
$$

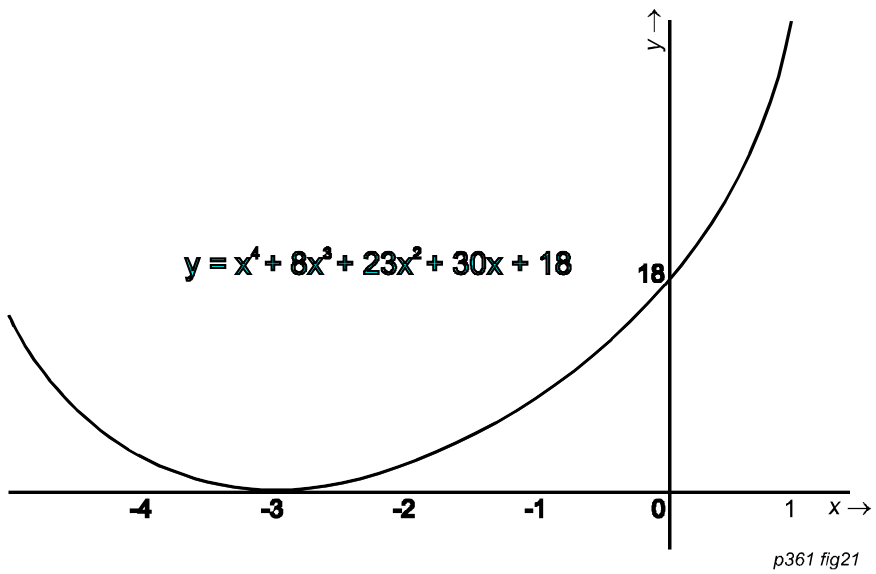
Hence $x^4 + 8x^3 + 23x^2 + 30x + 18 \equiv (x + 3)^2(x + 2x + 2)$

Now, $x^2 + 2x + 2 = 0$ has the roots

$$\frac{-2 \pm \sqrt{(4-8)}}{2} = \frac{-2 \pm \sqrt{(-4)}}{2} = -1 \pm \text{j}$$

Hence the roots are $x = -3, -3, -1+\text{j}, -1-\text{j}$.

A sketch of the graph is shown below for interest. Note that the graph touches the x axis at the repeated root.



SAQ2-7-4

The cubic polynomial $x^3 + 3x^2 + 9x - 13$ has one real factor $(x - 1)$ and 2 complex factors. Find all the factors and so write down the complete factorized expression.

Chapter 8

Solutions to SAQs

Solutions to SAQs

SAQ2-1-1

a.	$j^2 = -1$	b.	$j^3 = -j$	c.	$j^4 = 1$
d.	$-j^2 = 1$	e.	$(-j)^2 = -1$	f.	$j^5 = j$
g.	$j^6 = -1$	h.	$(-j)^4 = 1$	i.	$-j^4 = -1$

SAQ2-1-2

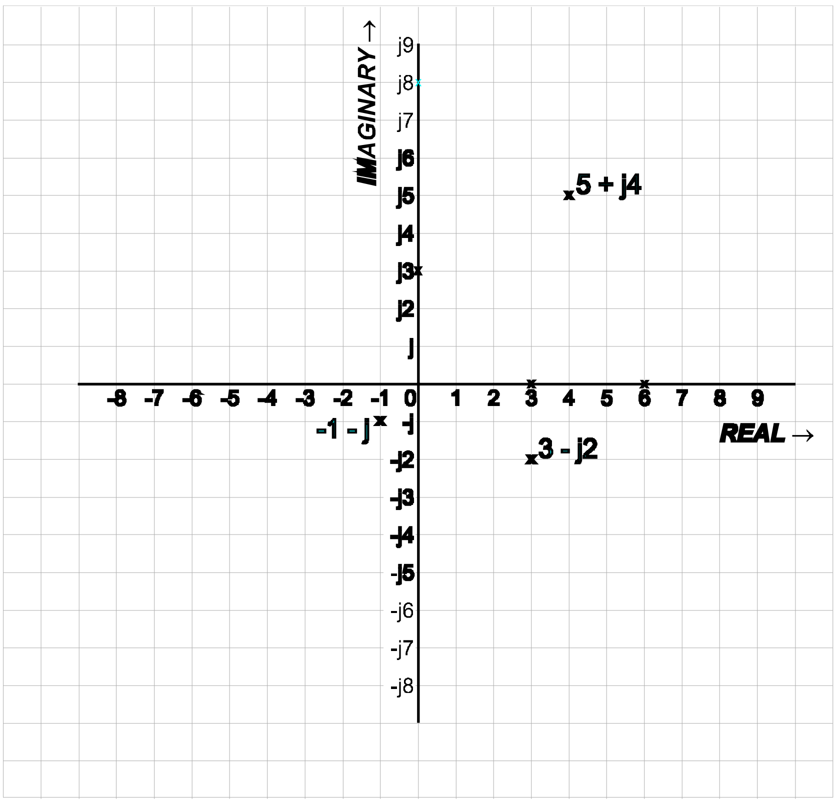
a.	j5	b.	–j6	c.	j6
d.	–4	e.	–2	f.	14
g.	–4	h.	–j8	i.	–16

SAQ2-1-3

a.

Number	Real part	Imaginary part
5 + j4	5	j4
3 – j2	3	–j2
–1 – j	–1	–j
6	6	0
j8	0	j8
√9	3	0
√–9	0	j3

b.



p361 fig22

SAQ2-2-1

	z_1	z_2	$z_1 + z_2$	$z_1 - z_2$
a.	5 + j2	3 + j4	8 + j6	2 – j2
b.	–3 + j	4 + j9	1 + j10	–7 – j8
c.	5 – j3	6 – j7	11 – j10	–1 + j4
d.	–3 + j2	89 – j10	5 – j8	–11 + j12
e.	–2 – j	–5 – j12	–7 – j13	3 + j11

SAQ2-2-2

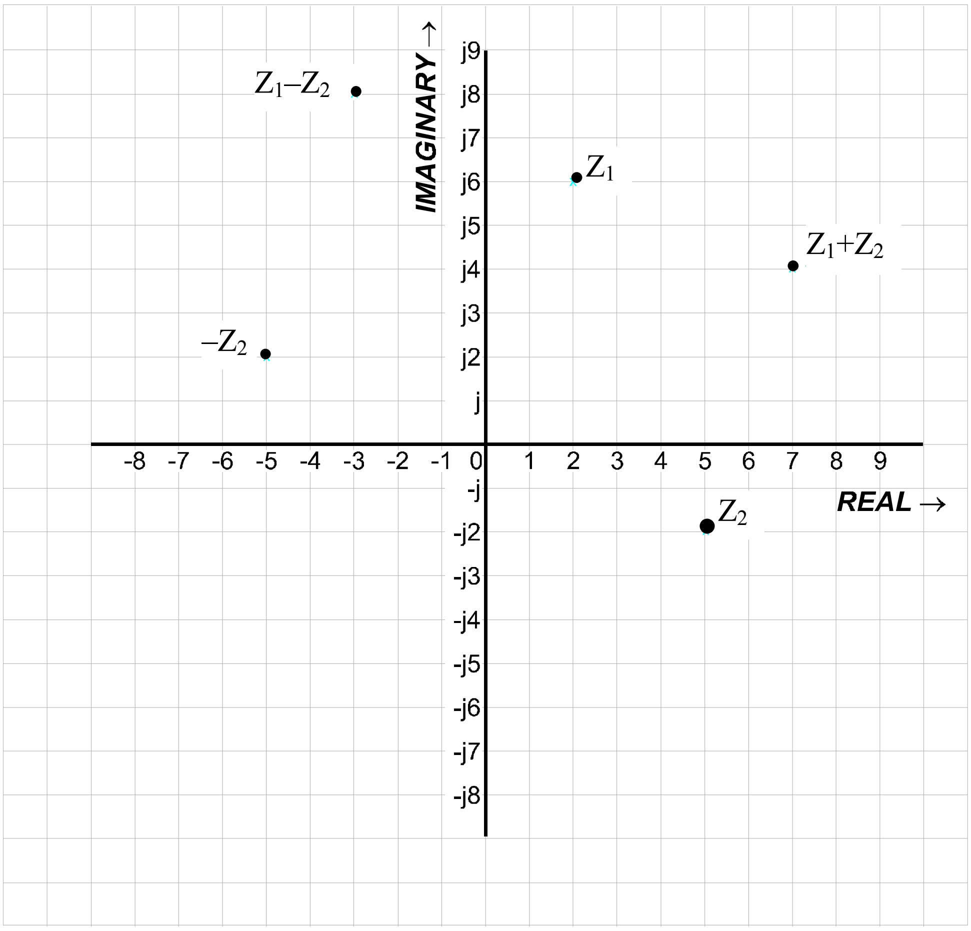
a. $z_1 = 2 + j6$

b. $z_2 = 5 - j2$

c. $-z_2 = -5 + j2$

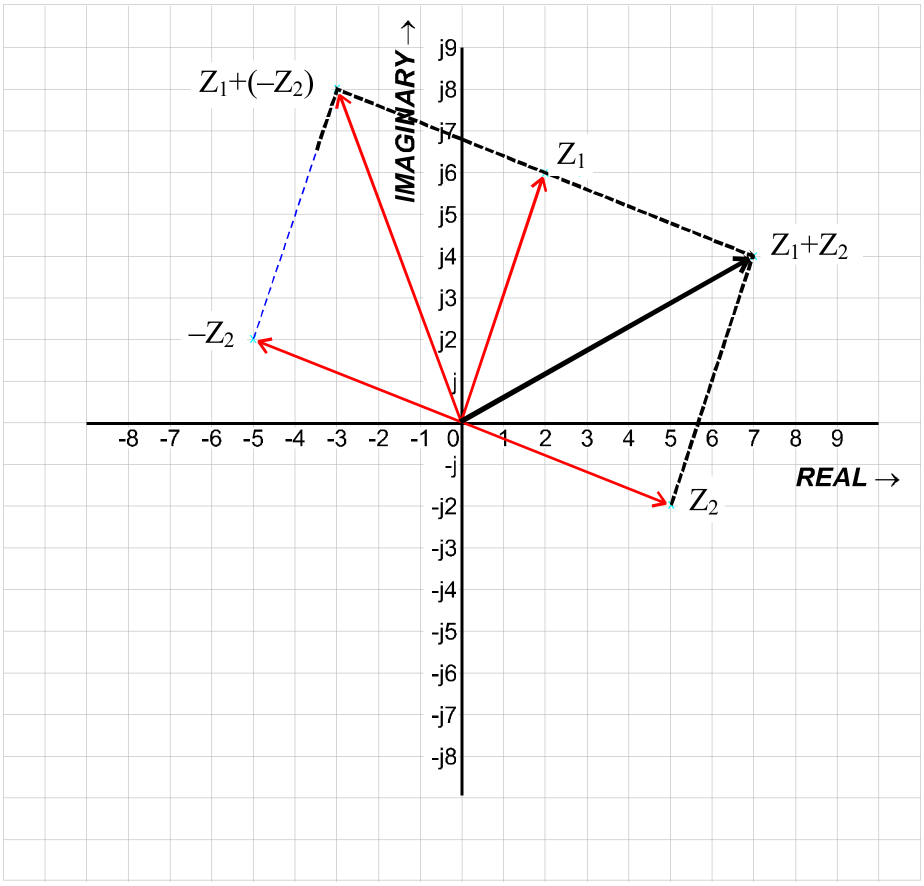
d. $z_1 + z_2 = 7 + j4$

e. $z_1 - z_2 = -3 + j8$



p361 fig23

SAQ2-2-2
Continued



p361 fig24

It can be seen that the vector $z_1 + z_2$ is the diagonal of the parallelogram constructed with z_1 and z_2.

Similarly $z_1 - z_2 = z_1 + (-z_2)$ is the diagonal of the parallelogram constructed with z_1 and $-z_2$.

This shows that the 2 methods give the same results.

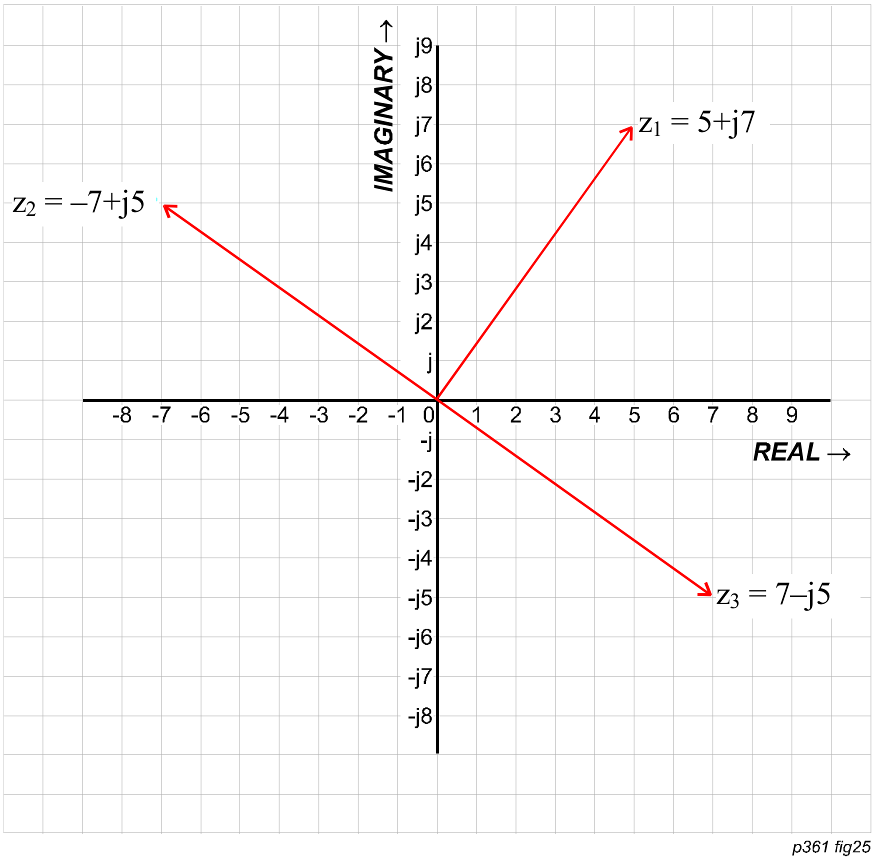
Note that $-z_2$ may be constructed by rotating z_2 through 180º.

SAQ2-2-3

	z_1	z_2	$z_1\ z_2$
a.	5 + j2	3 + j4	7 + j26
b.	–3 + j7	6 + *j8*	–74 + j18
c.	–4 – j	5 + j2	–18 – j13
d.	12 + j7	9 – j	115 + j51
e.	3 – j2	–4 – j5	–22 – j7
f.	–8 – j3	–3 – j5	9 + j49
g.	½ + j $^{\sqrt{3}}/_2$	½ + j $^{\sqrt{3}}/_2$	–½ + j $^{\sqrt{3}}/_2$

SAQ2-2-4

a.



b. $z_2 = \mathrm{j}\, z_1 = -7 + \mathrm{j}5$

c. The angle is 90º from z_1 to z_2, illustrating that multiplying any complex number by j gives a rotation of +90º.

d. $z_3 = -\mathrm{j}\, z_1 = 7 - \mathrm{j}5$

e. The angle is –90º (clockwise) from z_1 to z_3, illustrating that multiplying any complex number by –j gives a rotation of –90º.

The angle between z_2 and z_3 is 180º, since $z_3 = -z_2$, showing that a rotation of 180º gives the negative of a number.

SAQ2-2-5

z	z^*	$z + z^*$	$z\, z^*$
$4 + \mathrm{j}6$	$4 - \mathrm{j}6$	8	52
$3 - \mathrm{j}7$	$3 + \mathrm{j}7$	6	58
$-2 + \mathrm{j}5$	$-2 - \mathrm{j}5$	–4	29
$-9 - \mathrm{j}12$	$-9 + \mathrm{j}12$	–18	225
$2 - \mathrm{j}\sqrt{3}$	$2 + \mathrm{j}\sqrt{3}$	4	7
$1/\sqrt{2} + \mathrm{j}/\sqrt{2}$	$1/\sqrt{2} - \mathrm{j}/\sqrt{2}$	$\sqrt{2}$	1
$-\frac{1}{2} - \mathrm{j}\frac{\sqrt{3}}{2}$	$-\frac{1}{2} + \mathrm{j}\frac{\sqrt{3}}{2}$	–1	1

SAQ2-2-6

$-z^*$ represents a reflection in the **imaginary** axis.

SAQ2-2-7

a. $\dfrac{3 + j8}{1 + j} = \dfrac{(3 + j8)(1 - j)}{(1 + j)(1 - j)} = \dfrac{11 + j5}{2} = 5\cdot5 + j\,2\cdot5$

b. $\dfrac{5 - j6}{6 - j8} = \dfrac{(5 - j6)(6 + j8)}{(6 - j8)(6 + j8)} = \dfrac{78 + j4}{100} = 0\cdot78 + j0\cdot04$

c. $\dfrac{-8 - j7}{-7 - j} = \dfrac{(-8 - j7)(-7 + j)}{(-7 - j)(-7 + j)} = \dfrac{63 + j41}{50} = 1\cdot26 + j0\cdot82$

d. $\dfrac{10}{2 + j} = \dfrac{10(2 - j)}{(2 + j)(2 - j)} = \dfrac{20 - j10}{5} = 4 - j2$

e. $\dfrac{1}{R + j\omega L} = \dfrac{R - j\omega L}{(R + j\omega L)(R - j\omega L)} = \dfrac{R - j\omega L}{R^2 + \omega^2 L^2}$

$\dfrac{R}{R^2 + \omega^2 L^2} - j\dfrac{\omega L}{R^2 + \omega^2 L^2}$

SAQ2-2-8

$Z = (1000 + j250) + (2200 - j750) + (300 - j125)$ ohms

$+ \quad 3500 - j625$ ohms.

SAQ2-2-9

Working in kΩ

$$Z = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

$$= \frac{(1{\cdot}0 - j1{\cdot}5)(5{\cdot}0 + j3{\cdot}2)}{(1{\cdot}0 - j1{\cdot}5) + (5{\cdot}0 + j3{\cdot}2)}$$

$$= \frac{9{\cdot}8 - j4{\cdot}3}{6{\cdot}0 + j1{\cdot}7} = \frac{(9{\cdot}8 - j4{\cdot}3)(6{\cdot}0 - j1{\cdot}7)}{(6{\cdot}0 + j1{\cdot}7)(6{\cdot}0 - j1{\cdot}7)}$$

$$= \frac{51{\cdot}49 - j42{\cdot}46}{38{\cdot}89} = 1{\cdot}324 - j\,1{\cdot}092 \ \text{k}\Omega$$

SAQ2-2-10

$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}$$

$$= \frac{1}{2 + j3} + \frac{1}{1 - j} + \frac{1}{3 + j4}$$

$$= \frac{2 - j3}{13} + \frac{1 + j}{2} + \frac{3 - j4}{25}$$

(multiplying numerators and denominators by the conjugates)

$$= 1{\cdot}154 - j0{\cdot}231 + 0{\cdot}5 + j0{\cdot}5 + 0{\cdot}12 - j0{\cdot}16$$

$$= 0{\cdot}774 + j0{\cdot}109$$

$$Z = \frac{1}{0 \cdot 774 + j0 \cdot 109} = \frac{0 \cdot 774 - j0 \cdot 109}{0 \cdot 611}$$

$$= 1{\cdot}27 - j0{\cdot}18$$

SAQ2-2-11

$$\frac{3z}{1-j} + \frac{3z}{j} = \frac{4}{3-j}$$

$$3z\left[\frac{1}{1-j} + \frac{1}{j}\right] = \frac{4}{3-j}$$

$$3z\left[\frac{1+j}{2} - j\right] = \frac{4}{3-j}$$

$$3z(½ - j½) = \frac{4}{3-j}$$

$$\frac{3z}{2}(1-j) = \frac{4}{3-j}$$

$$3z/2 = \frac{4}{(3-j)(1-j)}$$

$$= \frac{4}{2-j4} = \frac{2}{1-j2}$$

$$= \frac{2(1+j2)}{5}$$

$$\therefore z = \frac{4(1+j2)}{15} = \frac{4}{15} + j\frac{8}{15}$$

SAQ2-3-1

a. $z = 6 + j8$

$r = \sqrt{6^2 + 8^2} = 10$

$\tan^{-1}(8/6) = 53{\cdot}1^{\circ}$. θ lies between 0 and 90º

$\therefore \theta = 53{\cdot}1^{\circ}$

Hence, $z = 10\angle 53{\cdot}1^{\circ}$

b. $z = -7 + j5$

$r = \sqrt{(-7)^2 + 5^2} = 8{\cdot}6$

$\tan^{-1}(-5/7) = -35{\cdot}5^{\circ}$. θ lies between 90º and 180º

$\therefore \theta = -35{\cdot}5^{\circ} + 180^{\circ} = 144{\cdot}5^{\circ}$

Hence, $z = 8{\cdot}6\angle 144{\cdot}5^{\circ}$

c. $z = -2{\cdot}5 - j3{\cdot}6$

$r = \; = \sqrt{(-2\cdot5)^2 + (-3\cdot6)^2} \;\; 4{\cdot}38$

$\tan^{-1}(3{\cdot}6/2{\cdot}5) = 55{\cdot}2^\circ$. θ lies between –90º and 180º.

$\therefore \theta = 55{\cdot}2^\circ - 180^\circ = -124{\cdot}8^\circ$

Hence, $z = 4{\cdot}38\angle-124{\cdot}8^\circ$

d. $z = 5 - j12$

$r = \sqrt{5^2 + (-12)^2} = 13$

$\tan^{-1}(-12/5) = -67{\cdot}4^\circ$. θ lies between 0 and –90º.

$\therefore \theta = -67{\cdot}4^\circ$

Hence, $z = 13\angle-67{\cdot}4^\circ$

SAQ2-3-2

a. $j2{\cdot}5 = 2{\cdot}5\angle90^\circ$

b. $-j7 = 7\angle-90^\circ$

c. $-5 = 5\angle180^\circ$

d. $3{\cdot}8 = 3{\cdot}8\angle0^\circ$

SAQ2-3-3

a. $z = 3 + j3$

$r = \sqrt{3^2 + 3^2} = 4{\cdot}24$

$\tan^{-1}(3/3) = \pi/4$. θ lies between 0 and π/2

$\therefore \theta = \pi/4$

Hence $z = 4{\cdot}24\angle\pi/4$

b. $z = -\sqrt{3} + j$

$r = \sqrt{\left(-\sqrt{3}\right)^2 + 1^2} = 2$

$\tan^{-1}(-1/\sqrt{3}) = -\pi 6$. θ lies between π/2 and π.

$\therefore \theta = -\pi/6 + \pi = 5\pi/6$

Hence $z = 2\angle 5\pi/6$

c. $z = -2 - j2\sqrt{3}$

$r = \sqrt{(-2)^2 + \left(-2\sqrt{3}\right)^2} = 4$

$\tan^{-1}(\sqrt{3}) = \pi/3$. θ lies between –π/2 and π.

$\therefore \theta = \pi/3 - \pi = -2\pi/3$

Hence, $z = 4\angle -2\pi/3$

SAQ2-3-4

a. $5\angle 32^\circ = (\cos 32^\circ + j\sin 32^\circ)$

$= 5(0{\cdot}8480 + j0{\cdot}5299) = 4{\cdot}24 + j2{\cdot}65$

b. $6{\cdot}2\angle 140^\circ = 6{\cdot}2(\cos 140^\circ + j \sin 140^\circ)$

$= 6{\cdot}2(-0{\cdot}7660 + j0{\cdot}6428) = -4{\cdot}75 + j3{\cdot}99$

c. $0{\cdot}8\angle -155^\circ = 0{\cdot}8(\cos -155^\circ + j \sin -155^\circ)$

$= 0{\cdot}8(-0{\cdot}9063 - j0{\cdot}4226) = -0{\cdot}73 - j0{\cdot}34$

d. $4{\cdot}9\angle -20^\circ = 4{\cdot}9(\cos -20^\circ + j \sin -20^\circ)$

$= 4{\cdot}9(0{\cdot}9397 - j0{\cdot}3420) = 4{\cdot}60 - j1{\cdot}68$

e. $3\angle \pi/4 = 3(\cos \pi/4 + j \sin \pi/4)$

$= 3(0{\cdot}7071 + j\,0{\cdot}7071) = 2{\cdot}12 + j2{\cdot}12$

SAQ2-3-5

a. $8\angle\pi/3 = 8(\cos\pi/3 + j\sin\pi/3)$

$= 8(0{\cdot}5 + j0{\cdot}8660 \quad = \quad 4{\cdot}00 + j6{\cdot}93$

b. $5\angle5\pi/6 = 5(\cos 5\pi/ + j\sin 5\pi/6)$

$= 5(-0{\cdot}8660 + j0{\cdot}5) \quad = \quad -4{\cdot}33 + j2{\cdot}50$

c. $\sqrt{2}\angle-\pi/4 = \sqrt{2}(\cos -\pi/4 + j\sin -\pi/4)$

$= \sqrt{2}(1/\sqrt{2} - j\,1/\sqrt{2}) \quad = \quad 1 - j$

d. $3\angle-\pi/2 = 3(\cos -\pi/2 + j\sin -\pi/2)$

$= 3(0 - j) \quad = \quad -j3$

e. $7{\cdot}5\angle\pi = 7{\cdot}5(\cos\pi + j\sin\pi)$

$= 7{\cdot}5(01 + j0) \quad = \quad -7{\cdot}5$

SAQ2-3-6

a. $3{\cdot}2\angle80^o \times 4{\cdot}5\angle23^o \quad = \quad 14{\cdot}4\angle103^o$

b. $7{\cdot}4\angle120^o \times 8\angle75^o \quad = \quad 59{\cdot}2\angle195^o = 59{\cdot}2\angle-165^o$

c. $8{\cdot}2\angle-\pi/6 \times 3{\cdot}5\angle2\pi/3 \quad = \quad 28{\cdot}7\angle\pi/2$

d. $9{\cdot}5\angle-40^o \times 3\angle-175^o \quad = \quad 28{\cdot}5\angle-215^o = 28{\cdot}5\angle145^o$

e. $2{\cdot}2\angle\pi \times 7{\cdot}4\angle\pi/4 \quad = \quad 16{\cdot}28\angle5\pi/4 = 16{\cdot}28\angle-3\pi/4$

	f.	$4{\cdot}8\angle 135^\circ \div 3{\cdot}2\angle 70^\circ = 15\angle 65^\circ$
	g.	$3{\cdot}28\angle 150^\circ \div 16{\cdot}4\angle -80^\circ = 0{\cdot}2\angle 230^\circ = 0{\cdot}2\angle -130^\circ$
	h.	$19\angle -100^\circ \div 2\angle 80^\circ = 9{\cdot}5\angle -180^\circ = 9{\cdot}5\angle 180^\circ$
	i.	$15\angle 3\pi/4 \div 4\angle -2\pi/3 = 3{\cdot}75\angle 17\pi/12 = 3{\cdot}75\angle -7\pi/12$
SAQ2-4-1	a.	$4{\cdot}5\angle 30^\circ = 4{\cdot}5\, e^{j\pi/6}$
	b.	$2{\cdot}5 - j1{\cdot}2 = 2{\cdot}77\angle -0{\cdot}4475 = 2{\cdot}77\, e^{-j0{\cdot}4475}$
	c.	$-10 - j12 = 15{\cdot}62\angle -2{\cdot}266 = 15{\cdot}62\, e^{-j2{\cdot}266}$
	d.	$2 + j2\sqrt{3} = 4\angle 1{\cdot}047 = 4e^{j1{\cdot}047}$

SAQ2-4-2

a. $(2\angle 20^o)^3 = 2^3\angle 3\times 20^o = 8\angle 60^o = 4{\cdot}00 + j6{\cdot}93$

b. $(3\angle -100^o)^4 = 3^4\angle 4\times -100^o = 81\angle -400^o = 81\angle -40$
$= 62{\cdot}05 - j52{\cdot}07$

c. $(2 + j3)^6 = (3{\cdot}606\angle 56{\cdot}31I)^6 = 3{\cdot}6\text{–}6^6\angle 6\times 56{\cdot}31^o$
$= 2197\angle 337{\cdot}86^o = 2197\angle -22{\cdot}14^o$
$= 2035 - j828$

d. $(-3 - j4)^5 = (5\angle -126{\cdot}87^o)^5 = 5^5\angle 5\text{a}\text{–}126{\cdot}87$
$= 3125\angle -634{\cdot}35^o = 3125\angle 85{\cdot}65^o$
$= 237 + j3116$

e. $(3\angle -\pi/3)^2 = {}^3 2\angle 2\times -\pi/3 = 9\angle -2\pi/3$
$= -4{\cdot}5 - j7{\cdot}79$

SAQ2-4-3

$$\tan x \equiv \frac{\sin x}{\cos x}$$

$$\equiv \frac{\frac{1}{2j}\left(e^{jx} - e^{-jx}\right)}{\tfrac{1}{2}\left(e^{jx} + e^{-jx}\right)}$$

$$\equiv \frac{(e^{jx} - e^{-jx})}{j(e^{jx} + e^{-jx})}$$

$$\equiv \frac{j(e^{-jx} - e^{jx})}{(e^{jx} + e^{-jx})}$$ Since $1/j = -j$

$$\equiv \frac{j(1 - e^{2jx})}{(1 + e^{2jx})}$$ multiplying top and bottom by e^{jx}

SAQ2-5-1

a. Principal root = $25^{\frac{1}{2}}\angle -120^\circ \div 2 = 5\angle -60^\circ$
$= 2{\cdot}5 - j4{\cdot}33$

Hence square roots are $2{\cdot}5 - j4{\cdot}33$ and $-2{\cdot}5 + j4{\cdot}33$

b. $4 - j12 = 13\angle -67{\cdot}38^\circ$

Principal square root is $13^{\frac{1}{2}}\angle -67{\cdot}38^\circ \div 2 = 3{\cdot}606\angle -33{\cdot}69$
$= 3 - j2$

Hence, square roots are $3 - j2$ and $-3 + j2$

c. $-24 - j70 = 74\angle -108{\cdot}92^\circ$

Principal square root is $74^{\frac{1}{2}}\angle -108{\cdot}92^\circ \div 2 = 8{\cdot}602\angle -54{\cdot}46^\circ$
$= 5 - j7$

Hence, square roots are $5 - j7$ and $-5 + j7$

d. $6 + j8 = 10\angle 53{\cdot}13^\circ$

Principal square root is $10^{\frac{1}{2}}\angle 53{\cdot}13^\circ \div 2 = 3{\cdot}162\angle 26{\cdot}57^\circ$
$= 2{\cdot}828 + j1{\cdot}414$

Hence, square roots are $2{\cdot}828 + j1{\cdot}414$ and $-2{\cdot}828 - j1{\cdot}414$

e. $-j9 = 9\angle -90^\circ$

Principal square root is $9^{\frac{1}{2}}\angle -90^\circ \div 2 = 3\angle -45^\circ$
$= 2{\cdot}121 - j2{\cdot}121$

Hence, square roots are $2{\cdot}121 - j2{\cdot}121$ and $-2{\cdot}121 + j2{\cdot}121$

SAQ2-5-2

a. One cube root is $125^{1/3}\angle{-150^\circ} \div 3$

$= 5\angle{-50^\circ} \quad = \quad 3{\cdot}21 - j3{\cdot}83$

The other roots are $5\angle(-50^\circ + 120^\circ) = 5\angle 70^\circ = 1{\cdot}71 + j4{\cdot}70$

and $5\angle(-50^\circ - 120^\circ) = 5\angle{-170^\circ} = -4{\cdot}92 - j0{\cdot}87$

b. $-610 - j182 = 636{\cdot}57\angle{-163{\cdot}39^\circ}$

One cube root is $636{\cdot}57^{1/3}\angle{-163{\cdot}39^\circ} \div 3 = 8{\cdot}602\angle{-54{\cdot}46^\circ}$

$= 5 - j7$

The other roots are $8{\cdot}602\angle(-54{\cdot}46^\circ + 120^\circ) = 8{\cdot}602\angle 65{\cdot}54^\circ$

$= 3{\cdot}56 + j7{\cdot}83$

and $8{\cdot}602\angle(-54{\cdot}46^\circ - 120^\circ) = 8{\cdot}602\angle{-174{\cdot}46^\circ}$

$= 8{\cdot}56 - j0{\cdot}83$

SAQ2-5-3

$-1 = 1\angle 180^\circ$

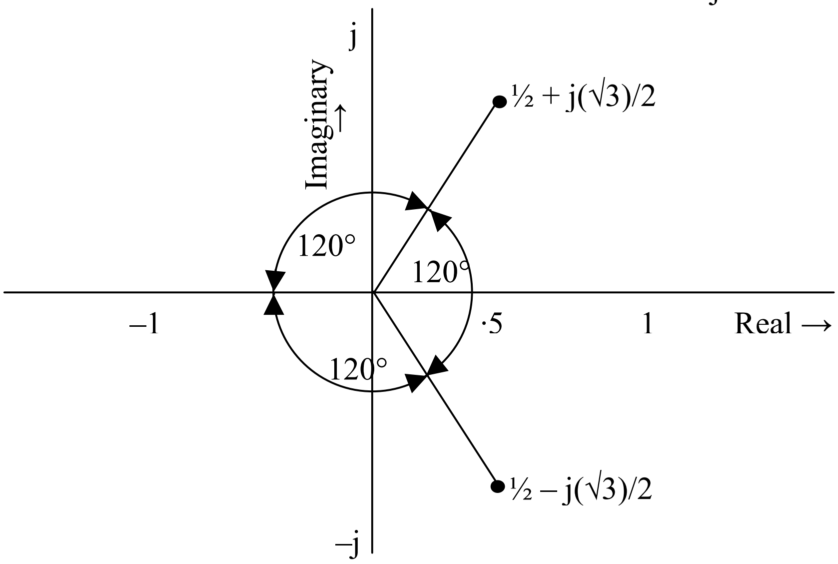
One cube root is $1^{1/3}\angle 180^\circ \div 3 = 1\angle 60^\circ = \tfrac{1}{2} + j$

$= 0{\cdot}5 + j0{\cdot}866$

The other cube roots are $1\angle(60^\circ + 120^\circ) = 1\angle 180^\circ = -1$

and $1\angle(60^\circ - 120^\circ) = 1\angle{-60^\circ} = \tfrac{1}{2} + j\,\frac{\sqrt{3}}{2}$

$= 0{\cdot}5 - j0{\cdot}866$



SAQ2-5-4

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

Substituting in the figures:

$R + j\omega L = 5 + j62{\cdot}83 = 63{\cdot}03\angle 85{\cdot}45^\circ$

$G + j\omega C = (2 + j18{\cdot}85) \times 10^{-6} = 18{\cdot}96 \times 10^{-6}\angle 83{\cdot}94^\circ$

$$\frac{R + j\omega L}{G + j\omega C} = \frac{63\cdot 03\angle 85\cdot 45^\circ}{18\cdot 96 \times 10^{-6}\angle 83\cdot 94^\circ} = 3{\cdot}324 \times 10^{6}\angle 1{\cdot}51^\circ$$

$\therefore Z_0 = \sqrt{3{\cdot}324 \times 10^{6}}\angle 1{\cdot}51^\circ \div 2 = 1{\cdot}823 \times 10^{3}\angle 0{\cdot}755^\circ$

$= 1823 + j24$ ohms.

SAQ2-5-5

$\gamma \sqrt{(R + j\omega L)(G + j\omega C)}$

Substituting in the figures

$R + j\omega L = 50 + j40{\cdot}21$

$G + j\omega = j2{\cdot}011 \times 10^{-7}$

$(R + j\omega L)(G + j\omega C) = j2{\cdot}011 \times 10^{-7}(50 + j40{\cdot}21)$

$= (-8{\cdot}08 + j10{\cdot}05) \times 10^{-6}$

$= 12{\cdot}90 \times 10^{-6}\angle 128{\cdot}8^\circ$

$\therefore \gamma = \sqrt{12{\cdot}90 \times 10^{-6}}\angle 128{\cdot}8^\circ \div 2 = 3{\cdot}59 \times 10^{-3}\angle 64{\cdot}4^\circ$

$= (1{\cdot}55 + j3{\cdot}24) \times 10^{-3}$

SAQ2-6-1

$(a + jb)^2 + b^2$

$= \quad a^2 - b^2 + 2ajb + b^2$

$= \quad a^2 + j2ab$

Hence, $a^2 + j2ab \quad = \quad 4 + j12$

Equating parts: $\quad a^2 = 4$
$\quad 2ab = 12$

Since a is positive, $\quad a = 2$.
Substituting in the second equation gives $b = 3$.

SAQ2-6-2

$$\frac{R_x - j/(\omega C_x)}{0{\cdot}1 - j/(\omega 3{\cdot}5\times10^{-6})} = \frac{24}{50}$$

$$= 0{\cdot}48$$

$\therefore\ R_x - j/(\omega C_x) \quad = \quad 0{\cdot}48\ \{0{\cdot}1 - j/(\omega 3{\cdot}5\times10^{-6})\}$

$= \quad 0{\cdot}048 - j0{\cdot}48/(\omega 3{\cdot}5\times10^{-6})$

Equating real parts: $\quad R_x = 0{\cdot}048$

Equating imaginary parts; $\quad 1/(\omega C_x) \quad = \quad 0{\cdot}48/(\omega 3{\cdot}5\times10^{-6})$

giving $\quad C_x \quad = \quad 3{\cdot}5\times10^{-6} \div 0{\cdot}48$

$= \quad 7{\cdot}3 \times 10^{-6}$

SAQ2-7-1

$$2x^2 + 12x + 50 = 0$$

Applying the formula for solution of quadratic equations;

$$x = \frac{-12 \pm \sqrt{(12^2 - 4\times2\times50)}}{2 \times 2}$$

$$= \frac{-12 \pm \sqrt{-256}}{4}$$

$$= \frac{-12 \pm j16}{4} = -3 \pm j4$$

Hence, the roots are $-3 + j4$ and $-3 - j4$.

SAQ2-7-2

$$3x^2 - 4x + 2 = 0$$

$$x = \frac{4 \pm\sqrt{[(-4)^2 - 4\times3\times2]}}{2 \times 3}$$

$$= \frac{4 \pm \sqrt{-8}}{6} = \frac{4 \pm 2\sqrt{-2}}{6}$$

$$= {}^2/_3 \pm j\,{}^1/_3 \sqrt{2} = 0{\cdot}67 \pm j0{\cdot}47$$

Hence, the roots to 2 decimal places are $0{\cdot}67 + j0{\cdot}47$ and $0{\cdot}67 - j0{\cdot}47$

SAQ2-7-3

a. $x^2 - 10x + 26 \equiv (x - \alpha)(x - \beta)$

where α, β are the roots of $x^2 - 10x + 26 = 0$.

Hence, $\alpha, \beta = \frac{10 \pm \sqrt{[(-10)^2 - 4\times1\times26]}}{2 \times 1}$

$$= \frac{10 \pm \sqrt{-4}}{2} = \frac{10 \pm j2}{2}$$

$$= 5 \pm j$$

Hence, $x^2 - 10x + 26 \equiv (x - 5 - j)(x - 5 + j)$

b. $9x^2 - 12x + 13 \equiv 9(x - \alpha)(x - \beta)$

where α, β are the roots of $9x^2 - 12x + 13 = 0$.

Hence, $\alpha, \beta = \dfrac{12 \pm \sqrt{[(-12)^2 - 4\times9\times13]}}{2 \times 9}$

$= \dfrac{12 \pm \sqrt{-324}}{18}$

$= \dfrac{12 \pm j18}{18} = {}^2/_3 \pm j$

Hence, $9x^2 - 12x + 13 \equiv 9(x - {}^2/_3 - j)(x - {}^2/_3 + j)$

$= (3x - 2 - j3)(3x - 2 + j3)$

c. $2x^2 + 8 \equiv 2(x - \alpha)(x - \beta)$

where α, β are the roots of $2x^2 + 8 = 0$.

$2(x^2 + 4) = 0 \quad \therefore x^2 + 4 = 0 \quad \therefore x^2 = -4$

The roots are $\pm j2$

Hence $2x^2 + 8 \equiv 2(x - j2)(x + j2)$

SAQ2-7-4

$$x^3 + 3x^2 + 9x - 13 \equiv (x - 1)(ax^2 + bx + c)$$

Dividing $x^3 + 3x^2 + 9x - 13$ by $x - 1$

$$\begin{array}{r l}
 & x^2 + 4x + 13 \\
\cline{2-2}
x-1 \,) & x^3 + 3x^2 + 9x - 13 \\
 & \underline{x^3 - x^2} \\
 & \quad 4x^2 + 9x - 13 \\
 & \quad \underline{4x^2 - 4x} \\
 & \qquad 13x - 13 \\
 & \qquad \underline{13x - 13} \\
 & \qquad \underline{\qquad\quad 0}
\end{array}$$

Hence, $x^3 + 3x^2 + 9x - 13 \equiv (x - 1)(x^2 + 4x + 13)$

$$\equiv (x - 1)\,(x - \alpha)(x - \beta)$$

Where $\alpha,\beta = \dfrac{-4 \pm \sqrt{(4^2 - 4\times1\times13)}}{2 \times 1}$

$$= \frac{-4 \pm \sqrt{-36}}{2} = \frac{-4 \pm j6}{2}$$

$$= -2 \pm j3$$

Hence, $x^3 + 3x^2 + 9x - 13 \equiv (x - 1)\,(x + 2 - j3)(x + 2 + j3)$