THE ROYAL SCHOOL OF SIGNALS

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MODULE 1 – ALGEBRA

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C O N T E N T S

Chapter 1

Revision

Revision

Introduction

The subject of **algebra** covers a very large range of different topics. The algebra we shall deal with in this course is that which has direct application to electrotechnology, and concerns the science of operations with numbers. Other types of algebra such as Boolean algebra are not covered. In this section we shall be considering operations with *real* numbers and *complex* numbers.

Basic concepts

Real and complex numbers are subject to the two basic algebraic operations of **addition** and multiplication represented by the operators + and × although for $a \times b$ we often write $a.b$ or simply ab . The basic laws are those with which we are familiar, i.e. the rules of simple arithmetic.

a. Commutative laws: $a + b = b + a$ and $ab = ba$

b. Associative laws: $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$

These statements may appear trivial since they are the rules of arithmetic that we all know, however, as we shall discover later, these rules do not necessarily apply to other structures such as vectors and matrices.

c. Distributive law: $a(b + c) = ab + ac$

i.e. multiplication distributes over addition. This is one of the axioms of numbers and may be verified by example, eg

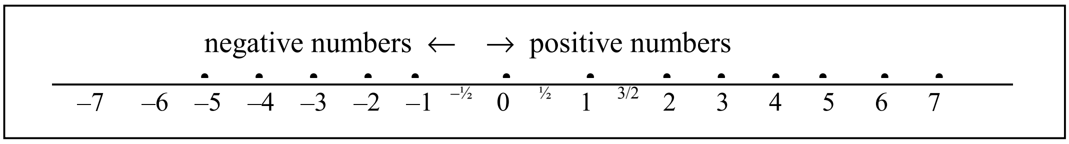
$$2 \times (3 + 4) = 2 \times 7 = 14$$

or $$2 \times (3 + 4) = 2 \times 3 + 2 \times 4 = 6 + 8 = 14$$

Real numbers

The first numbers which were conceived were the *natural numbers* or *positive integers,* i.e. 1, 2, 3, 4, ... the numbers used for counting. Eventually, in order to deal with practical problems, other numbers were introduced, such as zero, fractions and negative numbers.

A *rational* number is a fraction which can be expressed as the *ratio* of 2 integers, for example, ½, ¾, –¼ . The integers may also be regarded as rational numbers since an integer n may be considered to be $n/1$.



Graphical representation

The real numbers may be represented by points on a line either side of zero. This is sometimes called the Real line.

The number 0 (zero) has the unique properties: $a + 0 = a$, $a \times 0 = 0$

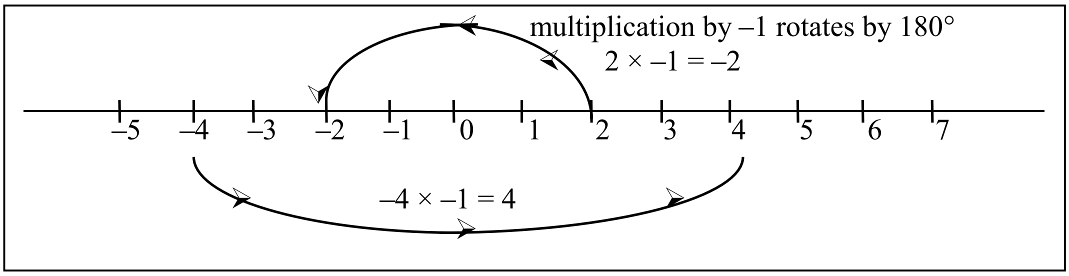
The number 1 (unity) has the unique property: $a \times 1 = a$

Subtraction

Subtraction is equivalent to moving in the negative direction, thus we may regard the operation of *subtraction* as being a special case of addition, i.e. the adding of a negative number: $a - b = a + (-b)$. Therefore we can see that the distributive law applies to subtraction in the same way as addition:

$$a(b - c) = ab - ac$$

Rotation



Multiplication by –1 represents a rotation of 180°.

Hence, $a \times (-1) = -a$, $-a \times (-1) = a$

Therefore $a \times (-b) = -ab$, $(-a) \times (-b) = ab$

Division

Every real number except zero, has an *inverse*. The inverse of the real number a is a number a^{-1} such that $a \times a^{-1} = 1$. We may regard the operation of *division* as being a special case of multiplication, i.e. multiplication by the inverse of a number: $a \div b = a \times b^{-1}$. Thus the inverse of a real number a may also be written as $1 \div a$ or $1/a$.

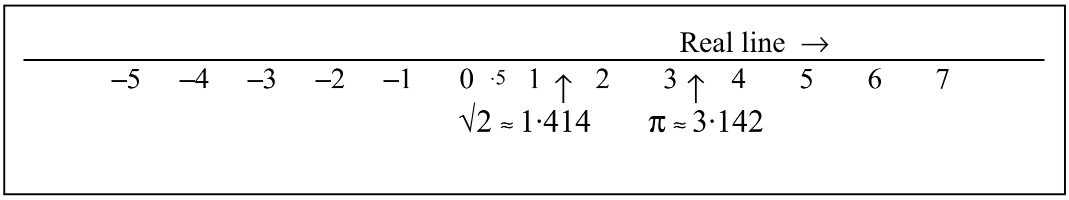
It follows that the distributive law applies to division in the same way as multiplication, eg

$$(a + b) \div c = (a + b) \times 1/c = a/c + b/c$$

As zero has no inverse, **division by zero is not possible** with real (or complex) numbers.

Irrational numbers

There are many real numbers which cannot be expressed as the ratio of two integers and are therefore called irrational. For example; $\sqrt{2}$, $\sqrt{3}$, π, e . Nevertheless they can be placed on the real line along with the rational numbers and can be evaluated as decimals to as many places as desired.



Infinity

The real numbers form a continuum from $-\infty$ to $+\infty$. The symbol ∞ denotes "infinity". It is important to understand that *infinity* cannot be regarded as a number since it does not obey the rules of algebra. In the algebra of real numbers, the statement $x = \infty$ has no meaning. We say that $x \to \infty$ which means that x approaches a very large value which is not measurable. We can also say:

As $x \to \infty$, $1/x \to 0$ i.e. as x becomes infinitely large, $1/x$ approaches zero, however, the expression $1 \div \infty$ has no meaning.

As $x \to 0$, $1/x \to \infty$, i.e. as x approaches zero, $1/x$ becomes infinitely large, however, the expression $1 \div 0$ has no meaning since division by zero is undefined.

The above two statements are examples of *limits* which will be discussed later in the course.

Equations

An equation consists of two expressions separated by the equality sign = The solution of algebraic equations is very simple if you remember that all you are doing is **arithmetic** with real numbers. Therefore, the only operations you need to perform as the operations of *addition, subtraction, multiplication*, and *division.* Sometimes other operations such as taking square roots may be required.

The statement [expression 1] = [expression 2] simply means that the 2 expressions have exactly the same numerical value, i.e. they are the same real number. Therefore any arithmetic operation performed on one side must be performed on the other side in order that the equality remain true.

Example Find the value of x which satisfies the equation:

$$2x + 3 = 11$$

This equation simply states that the number $(2x + 3)$ is equal to the real number 11. If we *subtract* 3 from each number we obtain

$$2x = 8$$

If we now *divide* each number by 2 we obtain $x = 4$, which is the only solution or *root* of this equation.

In solving equations you should avoid such vague rules as "take 3 to the other side and change its sign". There is no such operation in algebra. What is being performed is *subtraction*. If this point is appreciated, the student will have no difficulty in manipulating the most complicated expressions.

Identities

An equation in a variable x is satisfied by specific values of x only. For example, the above equation is satisfied by one single value of x, namely $x = 4$. The equation $x^2 = 9$ is satisfied by 2 real numbers only, $x = 3$ and $x = -3$. The equation $\sin x = 1$ is satisfied by an infinite set of numbers; $x = \pi/2 \pm 2n\pi$, where n is an integer.

An **identity** is satisfied by **all** real numbers. To distinguish an identity from an equation we use the symbol $\equiv$ which means *identically equal to.*

Examples

$$(x + 1)^2 \equiv x^2 + 2x + 1$$

This statement is true for every real value of x (also for every complex value). This may be checked by substituting any value you please in both sides and it will always be equal.

$\sin^2 x + \cos^2 x = 1$ is an **identity** as it is true for all values of x. Again, this may be verified by substituting any number for x. Therefore it is more informative to write it as:

$$\sin^2 x + \cos^2 x \equiv 1$$

SAQ1-1-1

If x is a **real number,** for what values of x are the following statements true?

a. $3(2x + 5) + 2 = 6x + 17$

b. $\dfrac{x - 1}{2} + \dfrac{4x + 1}{5} = x$

c. $x^2 + 4 = 0$

Application of distributive law

The distributive law is used to remove brackets.

eg $(a + b)(c + d)$

$= (a + b)c + (a + b)d$ applying the distributive law and treating $(a + b)$ as a single number.

$= ac + bc + ad + bd$ applying the distributive law to each bracket.

As a general rule of thumb, multiply everything in one bracket by everything in the other bracket.

EXAMPLE

Simplify $(2x - 3y)(4x + 7y) - xy + x^2 = 3y^2$

$(2x - 3y)(4x + 7y) - xy + x^2 = 3y^2$

$= 8x^2 + 14xy - 12xy - 21y^2 - xy + x^2 - 3y^2$

$= 9x^2 + xy - 24y^2$

Factorisation

To factorise expressions we apply the distributive law in reverse. For example

$2a^2b + 4a$

By inspection we see that both terms have a common factor of $2a$.

$\therefore\ 2a^2b + 4a = 2a(ab + 2)$

More complicated expressions may be factorised by grouping together terms which have common factors, for example

$15ac - 6a\mathrm{d} + 20bc - 8b\mathrm{d}$

$= 3a(5c - 2\mathrm{d}) + 4b(5c - 2\mathrm{d})$

$= (5c - 2\mathrm{d})(3a + 4b)$

The same result could have been obtained by grouping the pairs differently, eg

$15ac - 6a\mathrm{d} + 20b\mathrm{c} - 8b\mathrm{d}$

$= 5\mathrm{c}(3a + 4b) - 2\mathrm{d}(3a + 4b)$

$= (3a + 4b)(5\mathrm{c} - 2\mathrm{d})$

SAQ1-1-2

Applying the distributive law, simplify the following

a. $3(x + 2y) - 5(2x - y) + 12x - 2y$

b. $(a + 2b)(c - 3d) - ad + b(4c - 2d)$

c. $(2x + 3)(5x - 1) - x^2 - 3x + 1$

SAQ1-1-3

Factorise the following expressions

a. $3ab + 9bc$

b. $abc - a^2b + b^2c - ab^2$

c. $\frac{3\,ab}{4} - \frac{15\,a^2}{8}$

SAQ1-1-4

Solve the following equations for x .

a. $(2x + 3) - 4(x - 5) = 6$

b. $\frac{x}{3} + \frac{2x}{5} = \frac{22}{3}$

c. $3x + \frac{1}{4}x - \frac{3}{4}x + 5x = 1\frac{1}{2}$

Chapter 2

Indices and Logarithms

Indices

If a is a real number and n is a **positive integer** then we write

$$\underbrace{a \times a \times a \times \cdots \times a \times a}_{n \text{ times}} = a^n$$

This is said as "a raised to the power of n" . n is called the *index, power,* or *exponent*. a is called the *base.*

Rules of indices

Multiplication

If n and m are both positive integers we obtain the following rules.

$$a^m \times a^n = \underbrace{a \times a \times \cdots \times a}_{m \text{ times}} \quad \text{x} \quad \underbrace{a \times a \times \cdots \times a}_{n \text{ times}}$$

$$= \underbrace{a \times a \times a \times \cdots \times a \times a}_{m+n \text{ times}} \qquad = = a^{m+n}$$

Thus when multiplying powers of the same base, we **add** the indices.

It follows that $(a^m)^n = a^m \times \times^m \times a^m \times \cdots \times a^m = a^{\overbrace{m+m+\cdots+m}^{n \text{ times}}}$
$= a^{mn}$

Hence $(a^m)^n = (a^n)^m = a^{mn}$

Division

If $m>n$ then $a^m \div a^n = \dfrac{\overbrace{a \times a \times a \times \cdots \times a \times a}^{m \text{ times}}}{\underbrace{a \times a \times \cdots \times a \times a}_{n \text{ times}}}$

$$= \underbrace{a \times a \times \cdots \times a \times a}_{m-n \text{ times}} \qquad = a^{m-n}$$

Thus when dividing powers of the same base, we **subtract** the indices.

Examples

1. $10^2 \times 10^3 = 10\times10 \times 10\times10\times10 = 10^5 = 10^{2+3}$

2. $10^6 \div 10^4 = \dfrac{10\times10\times10\times10\times10\times10}{10\times10\times10\times10} = 10\times10 = 10^2$
$= 10^{6-4}$

Negative indices

If $m>n$ then $a^m \div a^n = \dfrac{\overbrace{a \times a \times \cdots \times a}^{m \text{ times}}}{\underbrace{a \times a \times a \times a \cdots \times a}_{n \text{ times}}}$

$$= \frac{1}{\underbrace{a \times a \times \cdots \times a \times a}_{n-m \text{ times}}}$$

$$= \frac{1}{a^{n-m}}$$

Now, $m–n$ is negative and from the above rule for division

$$a^m \div a^n = a^{m-n} = \frac{1}{a^{n-m}}$$

Hence $a^{-(n-m)} = = \dfrac{1}{a^{n-m}}$

So for a negative index, to keep the rules consistent, we define

$$a^{-p} = = \frac{1}{a^P}$$

Example

$$10^{-3} = \frac{1}{10^3} = 0{\cdot}001$$

The rules of multiplication are also consistent eg $10^4 \times 10^{-3} = 10^1 = 10$
i.e. the indices are added.

Zero index

By the rule for multiplication, $a^n \times a^0 = a^{n+0} = a^n$
By the rule for division $a^n \div a^n = a^{n-n} = a^0$

In both cases this implies that $a^0 = 1$

Rational indices

Consider the rational power n = ½

$$a^{½} \times a^{½} = a^{½+½} = a^1 = \text{a}$$

This implies that $a^{½}$ means $\sqrt{a}$, since $\sqrt{a} \times \sqrt{a} = a$.
Similarly, $= \underbrace{a^{1/n} \times a^{1/n} \times a^{1/n} \times \cdots \times a^{1/n}}_{n \text{ times}} = a$

Hence, $a^{1/n} = \sqrt[n]{a}$ i.e. the n^{th} root of a .

However, if n is an even number, $a^{1/n}$ only exists as a real number for $a \geq 0$, since even roots of negative numbers are "imaginary". We shall deal with imaginary numbers in Section 2: **Complex numbers.**

If m, n are integers, positive or negative, then $a^{m/n} = (a^m)^{1/n} = (a^{1/n})^m$

Hence $a^{m/n} = \sqrt[n]{(a^m)} = (\sqrt[n]{a})^m$

However, if ${}^m/_n$ is a fraction in its smallest form, and n is even, then $a^{m/n}$ can only be real for $a \geq 0$, since even roots of negative numbers are not real.

Examples

1. $8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$

   or $8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4$

2. $625^{-3/4} = 1/625^{3/4} = 1/(\sqrt[4]{615})^3 = 1/5^3 = 1/125 = 0{\cdot}008$

3. Express in its simplest form with positive indices

$$\tfrac{1}{2}(3x-2)^{3/4} - {}^3/_2(3x-2)^{-1/4}$$

$$= \frac{(3x-2)^{3/4}}{2} - \frac{3}{2(3x-2)^{1/4}}$$

$$= \frac{(3x-2)^{3/4}\,(3x-2)^{1/4} - 3}{2(3x-2)^{1/4}}$$

$$= \frac{(3x-2) - 3}{2(3x-2)^{1/4}} = \frac{3x-5}{2(3x-2)^{1/4}}$$

4. Simplify $\sqrt{\dfrac{b^{-3}\,(b^2)^{1/2}}{b^{-8}}}$

$$\left(\frac{b^{-3}b}{b^{-8}}\right)^{1/2} = \left(\frac{b^{-2}}{b^{-8}}\right)^{1/2} = (b^6)^{1/2} = b^3$$

Irrational indices	An irrational number cannot be expressed in the form *m/n*, therefore a^p where *p* is irrational cannot be defined as above. We shall leave this definition until after the next sub–section on *logarithms*.
SAQ1-2-1	Simplify, expressing the answer with positive indices.
	a. $\dfrac{1}{\sqrt{(z^3 \times z^{-5} \div z^{10})}}$
	b. $(x+2)^{½} - 4(x+2)^{-½} + 5(x+2)^{-3/2}$
SAQ1-2-2	Evaluate $6561^{-3/8}$, expressing the answer as a rational fraction.
SAQ1-2-3	If $a^{½} + a^{-½} = (2x+2)^{½}$, show that $x = ½(a + 1/a)$.

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Logarithms

The logarithm of a number to a given base is the power of the base which gives that number. i.e. $y = \log_b x$ if $x = b^y$.

Examples: $\log_{10}100 = 2$, since $100 = 10^2$

$\log_{10}1000 = 3$, since $1000 = 10^3$

$\log_{10} 0{\cdot}01 = -2$, since $0{\cdot}01 = 10^{-2}$

$\log_2 8 = 3$, since $8 = 2^3$

$\log_2 ¼ = -2$, since $¼ = 2^{-2}$

If b is any base, it follows that

$\log_b b = 1$, since $b^1 = b$

$\log_b 1 = 0$, since $b^0 = 1$

If $x<1$ then $\log x<0$

Standard notation

The most widely used logarithms are logarithms to the base ten. These are called *common logarithms.* Logarithms to the base e *(natural logarithms)* are often encountered in engineering. In information theory, logarithms to the base two *(binary logarithms)* are also used. Bases of logarithms are always positive constants. Bases other than 10, e, 2 are rarely used.

The IEC standard notation is as follows:

The common logarithm of x; $\log_{10}x$ is written $\lg x$
The natural logarithm of x; $\log_e x$ is written $\ln x$
The binary logarithm of x; $\log_2 x$ is written $\mathrm{lb}\, x$

The student should be aware that these conventions are not always followed. $\log x$ is often used to denote $\log_{10}x$ while in some pure mathematics texts, $\log x$ is used to denote $\log_e x$.

Antilogarithms

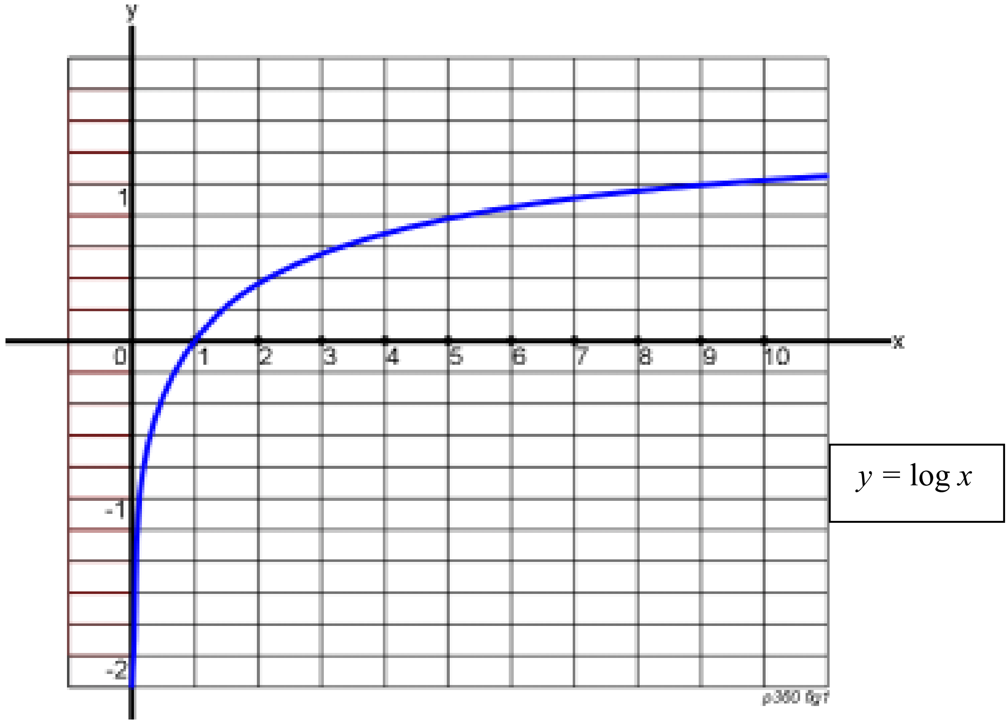
The inverse of a log is sometimes called the antilog. eg

$\log_{10}100 = 2 \;\therefore\; \text{antilog}_{10}2 = 100$

It is obvious that $\text{antilog}_{10}x \equiv 10^x$

Similarly $\text{antilog}_e x \equiv e^x$, $\text{antilog}_2 x = 2^x$

Graph of log function



It can be seen that as $x \to 0$, $y \to -\infty$ and that negative numbers have no logarithm. Since 10 is a positive constant, there is no real power to which it can be raised to give a negative result. Similarly, since e is a positive constant there is no real power to which it can be raised to give a negative result.

Rules of logarithms

As the log of a number is an index of a given base, the rules are the rules of indices.

$$\log_b(x\,y) = \log_b x + \log_b y$$

$$\log_b(x \div y) = \log_b x - \log_b y$$

$$\log(x^n) = n\log_b x$$

Examples

1. Given that lg 2 = 0·30103, lg 3 = 0·47712, using this information only find

a. lg 6 b. lg 9 c. lg 1·5

a. lg 6 = lg(3x2) = lg 3 + lg 2 = 0·30103 + 0·47712 = 0·77815
b. lg 9 = lg 3^2 = 2 lg 3 = 2 x 0·47712 = 0·95424
c. lg 1·5 = lg(3/2) = lg 3 – lg 2 = 0·47712 – 0·30103 = 0·17609

From the above rules it may be seen that $\log_b(1/x) = -\log_b x$
By the division rule; $\log_b(1/x) = \log_b 1 - \log_b x = 0 - \log_b x = -\log_b x$
By the power rule; $\log_b(1/x) = \log_b x^{-1} = -\log_b x$

Example

2. Given lg 2 = 0·30103, find lg 5 .

lg ½ = –lg 2 = –0·30103
lg 5 = lg(10 x ½) = lg 10 + lg ½
= 1 – 0·30103 = 0·69897

Change of base

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Proof Let $\log_a x$ = N then $x = a^N$
Taking logs to base b, $\log_b x$ = N $\log_b$a

Hence, $$N = \frac{\log_b x}{\log_b a}$$

$$\therefore \quad \log_a x = \frac{\log_b x}{\log_b a}$$

This rule may be used to find logs to base 2 using a calculator. eg

Example

$$\log_2 42 = \frac{\lg 42}{\lg 2} = \frac{1 \cdot 62325}{0 \cdot 30103} = 5{\cdot}3923$$

Alternatively, $$\log_2 42 = \frac{\ln 42}{\ln 2} = \frac{3 \cdot 73767}{0 \cdot 69315} = 5{\cdot}3923$$

Change of base for powers

It is often useful to express a power of one base in terms of a different base, usually as a power of e.

From the above rules, $\ln a^x = x \ln a$ (for a>0)

Taking antilogs to base e (raising e to the power of both sides)

$$a^x = \mathrm{e}^{x \ln a}$$

Irrational index

An irrational power may now be defined, i.e. $a^x = \mathrm{e}^{x \ln a}$ for irrational x. This definition only holds for $a > 0$, since $\ln a$ does not exist for $a \le 0$.

Examples

1. Express 10^3 as a power of e.

$10^3 = e^{3\ln(10)} = e^{3 \times 2\cdot3026} = e^{6\cdot9078}$

2. Express $(6e^{-2})^{1\cdot5}$ as a single exponent of e.

$(6e^{-2})^{1\cdot5} = 6^{1\cdot5}e^{-3} = e^{1\cdot5 \ln 6}\, e^{-3} = e^{2\cdot688}e^{-3} = e^{2\cdot688-3} = e^{-0\cdot312}$

3. Solve for x

a. $3^{2x-6} = 9$

b. $7^{2-x} = 2^{x+3}$

c. $3\log_2(3x+1) - \log_2(x-3)^3 = 12$

a. 9 is an exact power of 3, i.e. $3^{2x-6} = 3^2$
Taking logs to base 3; $2x - 6 = 2$, Hence $x = 4$

b. 7 and 2 cannot be expressed exactly in the same base, so taking logs to any base;

$$(2-x)\log 7 = (x+3)\log 2$$
$$2\log 7 - x\log 7 = x\log 2 + 3\log 2$$
$$2\log 7 - 3\log 2 = x(\log 2 + \log 7)$$
$$\log 7^2 - \log 2^3 = x(\log(2\times7))$$

$$x = \frac{\log 49 - \log 8}{\log 14}$$

Evaluating using common logs, $x = 0\cdot687$

c. This may be rewritten as $3\log_2(2x+1) - 3\log_2(x-3) = 12$

Divided by 3; $\log_2(2x+1) - \log_2(x-3) = 4$

This may be rewritten as $\log_2\left(\frac{2x+1}{x-3}\right) = 4$

Hence, $\frac{2x+1}{x-3} = 2^4$

$\therefore \quad 2x + 1 = 16x - 48$

$49 = 14x \qquad \therefore \quad x = 3\tfrac{1}{2}$

Example

4. Rearrange the following formula in terms of i.

$$t = \frac{L}{R} \ln\left(\frac{E}{E - iR}\right)$$

Rearranging

$$\frac{Rt}{L} = \ln\left(\frac{E}{E - iR}\right)$$

$$\therefore \quad e^{Rt/L} = \frac{E}{E - iR}$$

$$E - iR = E \div e^{Rt/L} = E\, e^{-Rt/L}$$

$$E(1 - e^{-Rt/L}) = iR$$

$$i = \frac{E}{R}(1 - e^{-Rt/L})$$

SAQ1-2-4

Prove the rules of logarithms, i.e.

a. $\log_b(x\,y) = \log_b x + \log_b y$

b. $\log_b(x \div y) = \log_b x - \log_b y$

c. $\log(x^n) = n \log_b x$

SAQ1-2-5

Given lg 5 = 0·69897, lg 7 = 0·84510, using this information only, find
(a) lg 2 (b) lg 14 (c) lg 3·5 (d) lg 1·96 (e) lg 8·75 , expressing the answers to 4 places of decimals.

SAQ1-2-6

Using the change of base rule for logarithms:

a. Evaluate $\log_2 50$ to 4 decimal places.

b. Prove $\log_a b = 1/(\log_b a)$, where a, b are any 2 numbers.

SAQ1-2-7

Solve for x without using log tables or calculator.

a. $\log_{10}x = \log_{10}3 + \log_{10}4 - \log_{10}6$

b. $2^{x+1} \,.\, 3^{x-1} - 2^{x} \,.\, 3^{x-2} = 120$

SAQ1-2-8

The voltage v on a transmission line at a distance x km from the source is given by $v = V_A e^{-ax}$ where V_A is the transmission voltage and α is the attenuation constant. If $V_A = 10$ volts, $\alpha = 0{\cdot}04$, find the distance from the source at which the voltage is 3·68 volts.

SAQ1-2-9 Rearrange the following formula to obtain an expression for t.

$$v = E(1 - e^{-t/CR})$$

SAQ1-2-10 Express 3×10^{-2} in the form e^{x}.

Chapter 3

Quadratics

Quadratics

A quadratic is a second degree polynomial. It is of the form $ax^2 + bx + c$, where $a, b, c,$ are constants.

Perfect Squares

<u>NOTE</u> $(x + a)(x - a) \equiv x^2 - a^2$ (Difference between 2 squares)

$(x + a)^2 \equiv x^2 + 2ax + a^2$ Perfect square

$(x - a)^2 \equiv x^2 - 2ax + a^2$ " "

In the expressions for perfect squares, the coefficient of x is $\pm 2a$ and the constant term is a^2. These are clearly related.

It can be seen that if a quadratic is a perfect square, then the constant term is the square of half the coefficient of x. eg

$$(x + 3)^2 \equiv x^2 + 6x + 9$$

$$(\tfrac{1}{2} \times 6)^2 = 9$$

$$(x - 5)^2 \equiv x^2 - 10x + 25$$

$$(\tfrac{1}{2} \times -10)^2 = 25$$

Completing the square

The above is used in completing the square, i.e. expressing a quadratic as the sum of a perfect square plus a constant. This process is often used when finding inverse Laplace transforms of second order expressions, applied to circuit analysis.

Examples

1.

Here we require $(\tfrac{1}{2} \times 6)^2 = 9$ to make a perfect square

take away 9 again to keep equal

$$\begin{aligned} x^2 + 6x + 13 &\equiv x^2 + 6x + 9 \quad -9 + 13 \\ &\equiv x^2 + 6x + 9 \quad + 4 \\ &\equiv (x + 3)^2 \quad + 4 \\ &\equiv (x + 3)^2 + 2^2 \end{aligned}$$

If the coefficient of x^2 is other than unity, take it out as a factor

2.

$$\begin{aligned} 2x^2 + 8x - 3 &\equiv 2\{ x^2 + 4x - {}^3/_2 \} \\ &\equiv 2\{ x^2 + 4x + 4 \quad - 4 - {}^3/_2 \} \\ &\equiv 2\{ (x+2)^2 - {}^{11}/_2 \} \\ &\equiv 2(x + 2)^2 - 11 \end{aligned}$$

(multiply back by 2)

3. $3x^2 - 6x + 10 \equiv 3\{ x^2 - 2x + {}^{10}/_3 \}$

$\equiv 3\{ x^2 - 2x + 1 \quad - 1 + {}^{10}/_3 \}$

$\equiv 3\{ (x - 1)^2 \quad + {}^{7}/_3 \}$

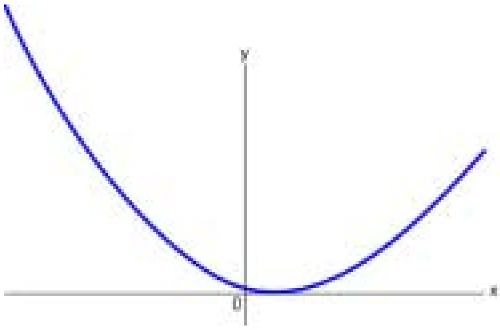
$\equiv 3\,(x - 1)^2 + 7$

SAQ1-3-1

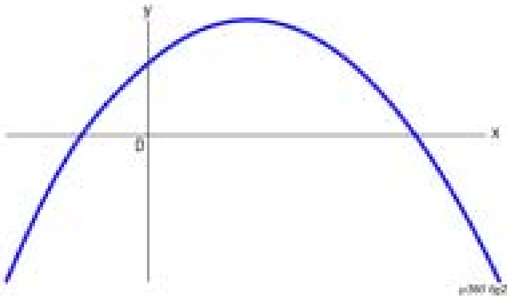
Express the following quadratics in the form $A(x + B)^2 + C$

a. $x^2 + 14x + 50$
b. $x^2 - 6x + 14$
c. $3x^2 + 15x - 4$
d. $2 + 6x - 2x^2$
e. $-5x^2 - 10x - 20$

Quadratic graphs



Graph of $y = ax^2 + bx + c$ for $a>0$



Graph of $y = ax^2 + bx + c$ for $a<0$

The graph of a quadratic function has a **parabolic** shape.

Quadratic equations

A quadratic equation is an equation of the form $ax^2 + bx + c = 0$.
Dividing through by a and completing the square:

$$a\{ x^2 + {}^b/_a\, x + {}^c/_a \} = 0$$

$$x^2 + {}^b/_a\, x + {}^c/_a = 0$$

$$(x + {}^b/_{2a})^2 - ({}^b/_{2a})^2 + {}^c/_a = 0$$

$$(x + {}^b/_{2a})^2 = {}^{b^2}/_{4a^2} - {}^c/_a$$

$$= \frac{b^2 - 4ac}{4a^2}$$

Square rooting both sides:

$$(x + {}^b/_{2a}) = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

We must put $\pm$ since every number has 2 square roots, one being the negative of the other.

Hence, $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This formula may be used to solve any quadratic equation.
A quadratic equation has 2 solutions for x which are known as *roots.*

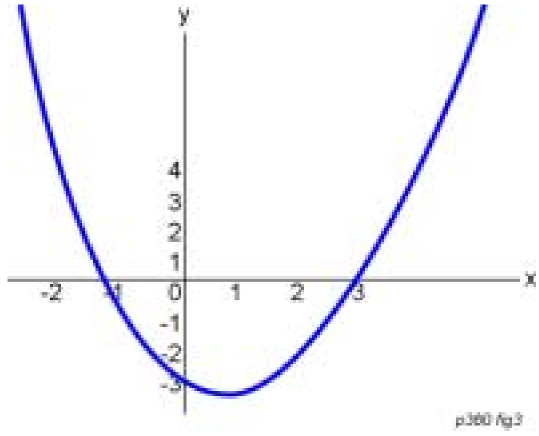
Discriminant

The quantity $b^2 - 4ac$ is called the *discriminant.*

Unequal roots

If $b^2 - 4ac > 0$ then the quadratic equation has 2 real roots which are unequal. In this case, the graph of $ax^2 + bx + c$ crosses the x axis, i.e. y = 0, in 2 points.

Example: The equation $x^2 - 2x - 3 = 0$ has the roots:



Graph of $y = x^2 - 2x - 3 \equiv (x+1)(x-3)$
$y = 0$ at the two points $x = -1$, $x = 3$

$$x = \frac{2 \pm \sqrt{\{(-2)^2 + 12\}}}{2}$$

$$= \frac{2 \pm \sqrt{16}}{2}$$

$$= 3\,, -1$$

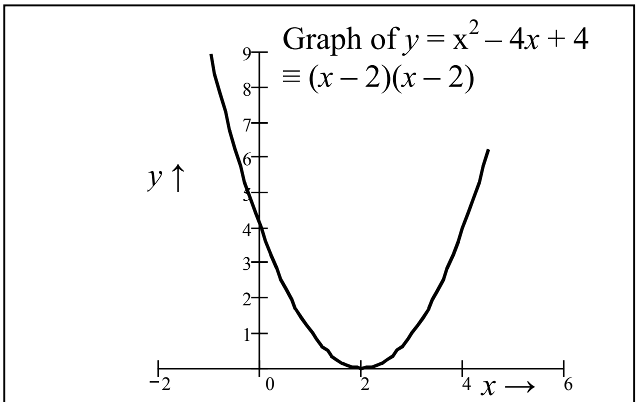
This can also be deduced from the factors $(x-3)(x+1) = 0$, giving

$x - 3 = 0$ or $x + 1 = 0$.

Equal roots

If $b^2 - 4ac = 0$ then we get the solution $x = -b/2a$. This is called a repeated root, i.e. the 2 roots are equal. In this case, the graph of $ax^2 + bx + c$ touches the x axis at one point only. Example:

The equation $x^2 - 4x + 4 = 0$ has the roots:



$y = 0$ at the point $x=2$

$$x = \frac{4 \pm \sqrt{0}}{2} = 2$$

Strictly speaking there are 2 roots which coincide.
i.e. $(x-2)(x-2) = 0$ gives the roots $x=2$, $x=2$.

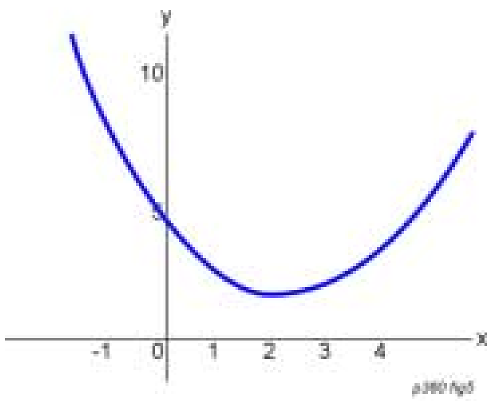
The nature of the roots of quadratic equations is important when we come to solve the second order differential equations which arise from the analysis of linear circuits.

Complex

If $b^2 - 4ac < 0$ then it has no real square root. In this case, the roots are complex.

roots | The graph of $ax^2 + bx + c$ does not touch or cross the x axis at any point.

Graph of y = log x



Graph of $y = x^2 - 4x + 5$
There is no real value of x for which $y = 0$

The complex roots of quadratic equations are also important in circuit theory and in the theory of filters.

We shall consider these further in *Section 2:* **Complex numbers.**

Factors of quadratics

Any quadratic expression may be resolved into 2 linear factors, i.e.

$$ax^2 + bx + c \equiv a\,(x - \alpha)(x - \beta)$$

Note the *minus* signs, so that putting x equal to α or β makes the expression zero.

Since α and β make $ax^2 + bx + c$ equal to zero, they must be the roots of the quadratic equation $ax^2 + bx + c = 0$

Example

$$2x^2 - 5x - 3 \;\equiv\; (x - 3)(2x + 1) \;\equiv\; 2(x - 3)(x + ½)$$

Obviously, $x = 3$ and $x = -½$ are the roots of the equation $2x^2 - 5x - 3 = 0$

When the factors involve simple numbers such as integers, the expression often may be factorised by inspection. In practical engineering problems the numbers are usually awkward decimals. The factors of a quadratic may be simply found by equating it to zero and solving the quadratic equation.

Example

Factorise $63x^2 - 103x - 26$

$63x^2 - 103x - 26 \equiv 63(x - \alpha)(x - \beta)$ where α, β are the roots of

$63x^2 - 103x - 26 = 0$

$$\therefore \alpha, \beta = \frac{103 \pm \sqrt{17161}}{126}$$

$$= \frac{13}{7} , \frac{-2}{9}$$

$$\therefore \quad 63x^2 - 103x - 26 \equiv 63(x - {}^{13}/_{7})(x + {}^{2}/_{9})$$

$$\equiv (7x - 13)(9x + 2)$$

In this example the factors are rational but in most practical cases this will not be so.

Irrational factors

Factorise $x^2 - 4x - 14$

$x^2 - 4x - 14 \equiv (x - \alpha)(x - \beta)$, where α, β are the roots of

$x^2 - 4x - 14 = 0 \qquad \therefore \alpha, \beta = \dfrac{4 \pm \sqrt{72}}{2} = \dfrac{4 \pm \sqrt{(36 \times 2)}}{2}$

$= \dfrac{4 \pm 6\sqrt{2}}{2} \qquad = 2 \pm 3\sqrt{2}$

Note that the irrational roots are conjugate surds.

$\therefore x^2 - 4x - 14 \equiv (x - 2 - 3\sqrt{2})(x - 2 + 3\sqrt{2})$

$= (x - 6{\cdot}24)(x + 2{\cdot}24)$ to 2 decimal places.

In most practical problems, a solution is obtained to a specified degree of accuracy.

Example

Factorise $6{\cdot}3x^2 + 1{\cdot}5x - 3{\cdot}2$

$6{\cdot}3x^2 + 1{\cdot}5x - 3{\cdot}2 \equiv 6{\cdot}3(x - \alpha)(x - \beta)$, where α, β are the roots of

$6{\cdot}3x^2 + 1{\cdot}5x - 3{\cdot}2 = 0$

$$\therefore \alpha, \beta = \frac{-1\cdot5 \pm \sqrt{82\cdot89}}{12\cdot6} = 0{\cdot}604 , -0{\cdot}842$$

$6{\cdot}3x^2 + 1{\cdot}5x - 3{\cdot}2 = 6{\cdot}3(x - 0{\cdot}604)(x + 0{\cdot}842)$ to 3 decimal places.

SAQ1-3-2

Find the factors of $2x^2 + 20x + 26$, expressing the answer in

(a) surd form (b) in decimal form to 3 decimal places.

SAQ1-3-3

Find the factors of $3x^2 - 2{\cdot}75x - 1{\cdot}2$, expressing the answer to 2 places of decimals.

Chapter 4

Algebraic division

Algebraic division

Polynomials

A polynomial in x is a function of x of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n$$

Where n is a non–negative integer and a_0 , a_1 , a_3 , ... are constants, some of which may be zero.
If a_n is not zero then n is the **degree** of the polynomial, i.e. the highest power present. For example $5 + 2x - 3x^2 + 7x^3$ is a polynomial of degree 3 (cubic polynomial).
As a special case, a constant a_0 may be regarded as a polynomial of degree zero, since $a_0 = a_0 x^0$.
Polynomials have applications to the theory of Error Detection and Correction in digital transmissions.

Addition and multiplication

The sum, difference, or product of two polynomials is clearly another polynomial. If P_m is a polynomial of degree m and P_n is a polynomial of degree n then $P_m \times P_n$ if a polynomial of degree $m+n$. If $m>n$ then $P_m \pm P_n$ is a polynomial of degree m .

Examples

$$(x^3 + x^2 - 4x + 2)(3x^2 - 2x + 1) \equiv 3x^5 + x^4 - 13x^3 + 15x^2 - 8x + 2$$

$$(x^3 + x^2 - 4x + 2) + (3x^2 - 2x + 1) \equiv x^3 + 4x^2 - 6x + 3$$

Polynomial division

However, when one polynomial is divided by another, the result is only a polynomial if it divides exactly with no remainder, i.e. if the smaller polynomial is a factor of the larger one.

$$\frac{2x^3 + 7x^2 + 5x - 2}{2x^2 + 3x - 1} = x + 2 \qquad \text{(divides exactly)}$$

$$\frac{4x^3 + x^2 - 4x + 2}{x^2 - 2x + 1} = 4x + 9 + \frac{10x - 7}{x^2 - 2x + 1}$$

We obtain a quotient of $4x + 9$ with a remainder of $10x - 7$.

These results may be obtained by a process of algebraic long division which consists of continually removing multiples of the divisor until either there is no remainder or the remainder is of lower degree than the divisor. To illustrate this process, let us remind ourselves of how we used to perform numerical long division before we had calculators.

Example

$2359 \div 15$

The number we are dividing by is called the *divisor* = 15
The number we are dividing is called the *dividend* = 2359

Dividing the dividend by the divisor gives a result called the *quotient* and a *remainder* which will be zero if the division is exact.

Applying the process of long division:

```
    157
15)2359     15 divides into 23 at most once, put 1 in the quotient.
   15       Multiply the divisor by 1, giving 15, put this underneath.
    85      Subtract, giving 8.  Bring down 5.  15 divides into 85 at most 5.
    75      Multiply divisor by 5, giving 75.
    109     Subtract, giving 10. Bring down the 9.  15 goes in at most 7.
    105     Multiply divisor by 7, giving 105.
      4     Subtract, giving 4 which is less than the divisor.
```

15 will not divide into 4 and so the process is finished. We have a quotient of 157 with a remainder of 4.

$$\text{Hence } 2359 \div 15 = 157\tfrac{4}{15}$$

This familiar process can be applied in exactly the same way to polynomial division. Taking the above two examples:

Examples of polynomial division

a. $\dfrac{2x^3+7x^2+5x-2}{2x^2+3x-1}$

```
                x + 2
2x²+3x−1)2x³+7x²+5x−2     2x² divides into 2x³, x times.  Put x in quotient.
         2x³+3x²−x        Multiply divisor by x then subtract.
             4x²+6x−2     2x² divides into 4x², 2 times.  Put 2 in quotient.
             4x²+6x−2     Multiply divisor by 2 then subtract.
                    0     Remainder is zero, so divides exactly.
```

The remainder is zero, therefore $\dfrac{2x^3+7x^2+5x-2}{2x^2+3x-1} = x+2$

b. $\dfrac{4x^3+x^2-4x+2}{x^2-2x+1}$

```
           4x + 9
x²−2x+1)4x³+x²−4x+2      x² divides into 4x³, 4x times.  Put 4x in quotient.
        4x³−8x²+4x       Multiply divisor by 4x then subtract.
            9x²−8x+2     x² divides into 9x², 9 times.  Put 9 in quotient.
           9x²−18x+9     Multiply divisor by 9 then subtract.
              10x−7      Remainder is of lower degree than divisor.
```

We are left with a remainder which is not divisible by $x^2 - 2x + 1$.
Hence, quotient = $4x + 9$, remainder = $10x - 7$.

i.e.. $\dfrac{4x^3+x^2-4x+2}{x^2-2x+1} = 4x+9+\dfrac{10x-7}{x^2-2x+1}$

If P_m is a polynomial of degree m and P_n is a polynomial of degree n, where $m \geq n$, then $P_m \div P_n$ consists of a quotient of degree $m–n$ with a remainder whose degree is less than n. the remainder may be zero.

SAQ1-4-1

Perform the following long divisions:

a. $(4x^4 - 4x^3 + 7x^2 - 3x - 4) \div (2x + 1)$

b. $(3x^4 + 2x^3 - 2x^2 - x + 6) \div (3x^2 - x + 2)$

c. $(2x^3 + 5x - 4) \div (x^3 + x^2 - 1)$

d. $\dfrac{x^3 + 3x^2 + 4x - 2}{(x+1)(x-3)}$

e. $(6x^3 + 11x^2 - x + 11) \div (3x + 7)$

This page has been left blank for the working of SAQs.

Chapter 5

Simultaneous Equations

Equations with more than one unknown

In chapter 1 we looked at the solution of equations with **one unknown**. An equation such as $\mathbf{2x - 3y = 6}$ has two unknown quantities, x and y. It is not possible to solve this equation uniquely for x or y. In order to solve it, we need some additional information. Simultaneous equations are a set of equations in more than one unknown, which may be solved to give values for all the unknowns.

This type of equation arises from problems in electrical networks which have more than one branch. Later, on the course at the Royal School of Signals, we shall solve this type of equation using matrix methods on a computer. Some pocket calculators have the facility for solving simultaneous equations but before you use this it is advisable to have some idea of how they are solved by hand.

Methods of solution

Method 1
Back-substitution

Consider the equation in line 2 above:

$$2x - 3y = 6$$

If we also had the information $x + y = 2$, then we can solve for both unknowns. This is a set of simultaneous equations.
We have :

$$2x - 3y = 6 \qquad ①$$
$$x + y = 2 \qquad ②$$

From equation ② $\qquad x = 2 - y$

Substituting this into equation ① :

$$2(2 - y) - 3y = 6$$

Solving this for y :

$$4 - 2y - 3y = 6$$
$$-5y = 2$$
$$y = -0{\cdot}4$$

Substituting the value of y back into equation ②

$$x = 2 - (-0{\cdot}4)$$
$$= 2{\cdot}4$$

The solutions are $x = 2{\cdot}4$, $y = -0{\cdot}4$

We could, of course, have substituted for y first, i.e. $y = 2 - x$ and then evaluated x.

Linear independence

To solve equations with **two** unknowns we require **two** equations.
To solve equations with **three** unknowns we require **three** equations.
To solve equations with ***n*** unknowns we require ***n*** equations.

Not only must we have ***n*** equations, these equations must be **linearly independent**. This means that any equation cannot simply be a linear combination of one or more of the other equations.

Example

Consider the equations

$$2x + 5y = 4 \qquad ①$$
$$6x + 15y = 12 \qquad ②$$

Equation ② is simply equation ① multiplied by 3. It contains no new information. Therefore it is not possible to solve for x and y uniquely.

Example

Consider the equations in 3 unknowns :

$$x + 3y - 4z = 2 \qquad ①$$
$$2x - y + 5z = 1 \qquad ②$$
$$4x + 5y - 3z = 5 \qquad ③$$

These equations are **not** linearly independent. Any one of these equations may be constructed from a linear combination of the other two, for example, equation ③ may be derived as equation ② + 2×equation ①.
Therefore it is not possible to solve for $x, y, z,$ uniquely.

The coefficients of $x, y, z,$ may be regarded as the vectors (1, 3, –4), (2, –1, 5), and (4, 5, –3). If these are not independent then the solution cannot be found.

SAQ 1-5-1

By the method of back-substitution, solve the following equations for x and y.

$$2x + 4y = 5$$
$$3x - 5y = 2$$

SAQ 1-5-2

A network gives rise to the following equations :

$$1{\cdot}5\, I_1 - 1{\cdot}2\, I_2 = 1{\cdot}8$$
$$2{\cdot}8\, I_1 - 8{\cdot}4\, I_2 = -5{\cdot}88$$

With the aid of a calculator, using back-substitution, solve for I_1 and I_2.

Method 2

In this method, one of the variables is firstly eliminated by performing a series of

Elimination

linear operations between the equations. Any equation or a multiple of it may be added to or subtracted from any other equation.

Example

Consider the equations that we solved above.

$$2x - 3y = 6 \quad ①$$
$$x + y = 2 \quad ②$$

Multiply equation ② by 2 giving equation ③

②×2 : $2x + 2y = 4$ ③

$2x - 3y = 6$ ①

Subtract equation ① from equation ③

③–① : $5y = -2$

$\therefore y = -0{\cdot}4$

We can now either back-substitute to find x or use elimination again. In this example, back-substitution is the simpler method, however to illustrate the point, now multiply equation ② by 3 giving equation ④.

$2x - 3y = 6$ ①

②×3 : $3x + 3y = 6$ ④

Now add equations ① and ④.

①+④ : $5x = 12$

$\therefore x = 2{\cdot}4$

Example

Solve for x and y.

$$3x + 2y = 25 \quad ①$$
$$2x - 5y = 4 \quad ②$$

Multiply equation ① by 2 and equation ② by 3

①×2 : $6x + 4y = 50$ ③
②×3 : $6x - 15y = 12$ ④

Subtract equation ④ from equation ③.

③–④ : $19y = 38$
$\therefore y = 2$

Back-substituting into equation ①
$$3x + 4 = 25$$
$$3x = 21$$
$$x = 7$$

SAQ 1-5-3

Solve by elimination

$$5x + 3y = 81$$
$$4x - 2y = 34$$

SAQ 1-5-4

Solve by elimination with the aid of a calculator

$$2{\cdot}5\, I_1 + 1{\cdot}2\, I_2 = 7{\cdot}56$$
$$3{\cdot}2\, I_1 - 0{\cdot}7\, I_2 = 6{\cdot}77$$

Equations in

The same variable is eliminated between two pairs of equations leaving two three unknowns

equations in two variables. These are then solved as described above. The third variable can then be obtained by back-substitution.

Example

Solve for *x, y, z.*

$$2x + 3y - 5z = 28 \quad ①$$
$$3x - 2y + 6z = -4 \quad ②$$
$$x + 4y + 3z = 21 \quad ③$$

Firstly choose which variable you wish to eliminate. This will depend upon which choice results in the simplest arithmetic. Suppose we choose *x.*

Multiply equation ③ by 2.

③×2 : $2x + 8y + 6z = 42$ ④
Subtract equation ① $2x + 3y - 5z = 28$ ①
④–①: $5y + 11z = 14$ ⑤

We now eliminate the **same** variable between **any other pair** of equations.
Multiply equation ③ by 3.
③×3 : $3x + 12y + 9z = 63$ ⑥
Subtract equation ② $3x - 2y + 6z = -4$ ②
⑥–② : $14y + 3z = 67$ ⑦

We now have the pair of equations in *y* and *z* :

$$5y + 11z = 14 \quad ⑤$$
$$14y + 3z = 67 \quad ⑦$$

Multiply equation ⑤ by 3 and equation ⑦ by 11
⑤×3 : $15y + 33z = 42$ ⑧
⑦×11 : $154y + 33z = 737$ ⑨
Subtract equation ⑧ from equation ⑨
⑨–⑧ : $139y = 695$

$$\therefore y = 5$$

Back-substituting in ⑤ gives $25 + 11z = 14$

$$\therefore z = -1$$

Back-substituting in ③ for both *y* and *z* gives

$$x + 20 - 3 = 21$$
$$\therefore x = 4$$

Solutions are $x = 4$, $y = 5$, $z = -1$.

SAQ 1-5-5

Solve for *x, y, z* :

$$x + y - z = 6 \quad ①$$
$$3x + 2y + 4z = 1 \quad ②$$
$$x - y + 2z = -6 \quad ③$$

SAQ 1-5-6

In a network the currents in three branches are related by the following equations :

$$3I_1 + 2I_2 + 6I_3 = 27{\cdot}5$$
$$3I_1 + 4(I_2 - I_3) = 2{\cdot}5$$
$$6(I_3 - I_1) + 4I_2 = 19$$

Rearrange these equations into a suitable form and solve them to find values for I_1, I_2, I_3.

Higher order equations

To solve four equations in four unknowns we progressively reduce it to three unknowns and then to two unknowns as before. This is obviously very laborious, and computer methods were developed many years ago to solve very large sets of simultaneous equations which arise commonly in engineering, initially using analogue computers and subsequently digital computers. It should also be appreciated that in real life problems, the numbers are not all nice round integers.

On your course at the Royal School of Signals, you will learn computer techniques for solving equations of this type.

Chapter 6

Solutions to SAQs

Solutions to SAQs

SAQ1-1-1 solution

a. Applying the distributive law to the left hand side,
$3(2x + 5) + 2 = 6x + 15 + 2 = 6x + 17$. This is identically equal to the right hand side of the statement, therefore the statement is true for **all** values of x . i.e. it is an identity.

b. Multiplying both sides by 10

$5(x-1) + 2(4x+1)$	=	$10x$	
$5x - 5 + 8x + 2$	=	$10x$	*(applying distributive law)*
$13x - 3$	=	$10x$	*(adding like terms)*
$3x$	=	3	*(subtract 10x and add 3)*
x	=	1	*(divide by 3)*

The statement is true for $x = 1$ only.

c. Subtracting 4 from both sides

$x^2 = -4$

There is **no real number** x for which this statement is true, since $x^2 \geq 0$ for all real x .

SAQ1-1-2 solution

a. $3(x + 2y) - 5(2x - y) + 12x - 2y$

$= 3x + 6y - 10x + 5y + 12x - 2y$

$= 5x + 9y$

b. $(a + 2b)(c - 3d) - ad + b(4c - 2d)$

$= ac + 2bc - 3ad - 6bd - ad + 4bc - 2bd$

$= ac + 6bc - 4ad - 8bd$

c. $(2x + 3)(5x - 1) - x^2 - 3x + 1$

$= 10x^2 + 15x - 2x - 3 - x^2 - 3x + 1$

$= 9x^2 + 10x - 2$

SAQ1-1-3 solution

a. $3ab + 9bc = 3b(a + 3c)$

b. $abc - a^2b + b^2c - ab^2$

$= b(ac - a^2 + bc - ab)$

$= b(a(c - a) + b(c - a))$

$= b(a + b)(c - a)$

c. $\frac{3ab}{4} - \frac{15a^2}{8}$

$= \frac{3a}{4}\left(b - \frac{5a}{2}\right)$

$= \frac{3a}{8}(2b - 5a)$

SAQ1-1-4 solution

a. $2(x + 3) - 4(x - 5) = 6$

$2x + 6 - 4x + 20 = 6$

$-2x + 26 = 6$

$20 = 2x$

$x = 10$

b. $\frac{x}{3} + \frac{2x}{5} = \frac{22}{x}$

Multiplying both sides by the lowest common denominator, 15

$5x + 6x = 110$

$11x = 110$

$x = 10$

SAQ1-1-4 solution

c. $3x + \frac{1}{4}x - \frac{3}{4}x + 5x = 1\frac{1}{2}$

$7\frac{1}{2}x = 1\frac{1}{2}$

$\frac{15x}{2} = \frac{3}{2}$

$15x = 3$

$x = {}^{1}/_{5}$

SAQ1-2-1 solution

a. $\frac{1}{\sqrt{(z^3 \times z^{-5} \div z^{10})}} = \frac{1}{(z^{-12})^{½}} = \frac{1}{z^{-6}} = z^6$

b. $(x+2)^{½} - 4(x+2)^{-½} + 5(x+2)^{-3/2}$

$= \sqrt{x+2} - \frac{4}{\sqrt{x+2}} + \frac{5}{\left(\sqrt{x+2}\right)^3}$

$= \frac{\{\sqrt{(x+2)}\}^4 - 4\{\sqrt{(x+2)}\}^2}{\{\sqrt{(x+2)}\}^3} + 5$

$= \frac{(x+2)^2 - 4(x+2) + 5}{\left(\sqrt{x+2}\right)^3} = \frac{x^2 + 1}{(x+2)^{3/2}}$

SAQ1-2-2 solution

$6561^{-3/8} = 1/(\sqrt[8]{6561})^3 = 1/3^3 = 1/27$

SAQ1-2-3 solution

$a^{½} + a^{-½} = (2x+2)^{½}$

Squaring both sides, $a + a^{-1} + 2 = 2x + 2$

$a + a^{-1} = 2x$

$\therefore \quad x = ½(a + 1/a)$

SAQ1-2-4 solution

a. Let $u = \log_b x$, $v = \log_b y$

Then $x = b^u$, $y = b^v$

$\therefore xy = b^u \; b^v$

$= b^{u+v}$

Hence, $\log_b(xy) = u + v$

$= \log_b x + \log_b y$

b. From above, also $x \div y = b^u \div b^v$

$= b^{u-v}$

Hence, $\log_b(x \div y) = u - v$

$= \log_b x - \log_b y$

c. From above, $u = \log_b x$ $\therefore x = b^u$

$\therefore x^n = (b^u)^n$

$= b^{nu}$

Hence, $\log_b(x^n) = nu$

$= n \log_b x$

SAQ1-2-5 solution

a. $\lg 2 = \lg(10 \div 5) = \lg 10 - \lg 5 = 1 - 0{\cdot}69897 = 0{\cdot}30103$

b. $\lg 14 = \lg(7 \times 2) = \lg 7 + \lg 2 = 0{\cdot}84510 + 0{\cdot}30103 = 1{\cdot}14613$

c. $\lg 3{\cdot}5 = \lg(7 \div 2) = \lg 7 - \lg 2 = 0{\cdot}84510 - 0{\cdot}30103 = 0{\cdot}54407$

d. $\lg 1{\cdot}96 = \lg(14^2 \div 100) = 2\lg 14 - \lg 100 = 2 \times 1{\cdot}14613 - 2 = 0{\cdot}29226$

e. $\lg 8{\cdot}75 = \lg(7 \times 5 \div 4) = \lg 7 + \lg 5 - 2\lg 2 = 0{\cdot}84510 + 0{\cdot}69897 - 0{\cdot}60206$
$= 0{\cdot}94201$

SAQ1-2-6 solution

a. $\log_2 50 = \dfrac{\log_{10} 50}{\log_{10} 2} = 5{\cdot}6439$

b. $\log_a b = \dfrac{\log_b b}{\log_b a} = 1/(\log_b a)$

SAQ1-2-7 solution

a. $\log_{10} x = \log_{10} 3 + \log_{10} 4 - \log_{10} 6$

$= \log_{10}(3 \times 4 \div 6) = \log_{10} 2$

$\therefore\ x = 2$

b. $2^{x+1}.3^{x-1} - 2^x.3^{x-2} = 120$

$2^x.2\ .\ 3^x.3^{-1} - 3^x.3^x.3^{-2} = 120$

${}^2/_3\,(2 \times 3)^x - {}^1/_9\,(2 \times 3)^x = 120$

$6.6^x - 6^x = 1080$

$5.6^x = 1080$

$6^x = 216 \qquad \therefore\ x = 3$

SAQ1-2-8 solution

$$v = V_A e^{-\alpha x}$$

Substituting in values; $3{\cdot}68 = 10\,e^{-0.04x}$

$$e^{-0{\cdot}04x} = 0{\cdot}368$$

Taking logs to base *e*; $-\,0{\cdot}04x = \ln 0{\cdot}368$

$$x = (\ln 0{\cdot}368) \div (-0{\cdot}04)$$

$$x = 25 \text{ km}$$

SAQ1-2-9 solution

$$v = E(1 - e^{-t/CR})$$

$$v/E = 1 - e^{-t/CR}$$

$$e^{-t/CR} = 1 - v/E = \frac{E - v}{E}$$

Inverting; $e^{t/CR} = \dfrac{E}{E - v}$

Taking logs; $t/CR = \ln\left(\dfrac{E}{E - v}\right)$

$$t = CR \ln\left(\frac{E}{E - v}\right)$$

SAQ1-2-10 solution

$$3 \times 10^{-2} = e^{\ln 3}\ e^{-2\ln 10}$$

$$= e^{\ln 3 - 2\ln 10}$$

$$= e^{-3{\cdot}5066}$$

SAQ1-3-1 solution

a. $x^2 + 14x + 50 \equiv x^2 + 14x + 49 - 49 + 50$

$\equiv (x + 7)^2 + 1$

b. $x^2 - 6x + 14 \equiv x^2 - 6x + 9 - 9 + 14$

$\equiv (x - 3)^2 + 5$

c. $3x^2 + 15x - 4 \equiv 3\{x^2 + 5x - {}^4/_3\}$

$\equiv 3\{x^2 + 5x + {}^{25}/_4 - {}^{25}/_4 - {}^4/_3\}$

$\equiv 3\{(x + {}^5/_2)^2 - {}^{91}/_{12}\}$

$\equiv 3(x + {}^5/_2)^2 - {}^{91}/_4$

d. $2 + 6x - 2x^2 \equiv -2\{x^2 - 3x - 1\}$

$\equiv -2\{x^2 - 3x + {}^9/_4 - {}^9/_4 - 1\}$

$\equiv -2\{(x - {}^3/_2)^2 - {}^{13}/_4\}$

$\equiv -2(x - {}^3/_2)^2 + {}^{13}/_2$

e. $-5x^2 - 10x - 20 \equiv -5\{x^2 + 2x + 4\}$

$\equiv -5\{x^2 + 2x + 4\}$

$\equiv -5\{(x + 1)^2 + 3\}$

$\equiv -5(x + 1)^2 - 15$

SAQ1-3-2 solution

$2x^2 + 20x + 26 \equiv 2(x - \alpha)(x - \beta)$

where α, β are the roots of $2x^2 + 20x + 26 = 0$

$$\therefore \alpha, \beta = \frac{-20 \pm \sqrt{(400 - 208)}}{4} = \frac{-20 \pm \sqrt{192}}{4}$$

$$= \frac{-20 \pm \sqrt{(64x3)}}{4} = \frac{-20 \pm 8\sqrt{3}}{4} = -5 \pm 2\sqrt{3}$$

$$\therefore 2x^2 + 20x + 26 \equiv 2(x + 5 - 2\sqrt{3})(x + 5 + 2\sqrt{3})$$

$$\equiv 2(x + 1{\cdot}536)(x + 8{\cdot}464)$$

SAQ1-3-3 solution

$3x^2 - 2{\cdot}75x - 1.2 \equiv 3(x - \alpha)(x - \beta)$

where α, β are the roots of $3x^2 - 2{\cdot}75x - 1{\cdot}2 = 0$

$$\therefore \alpha, \beta = \frac{2 \cdot 75 \pm \sqrt{(7.5625 + 14 \cdot 4)}}{6} = \frac{2 \cdot 75 \pm 4 \cdot 686}{6}$$

$$= 1{\cdot}25, -0{\cdot}32$$

$\therefore 3x^2 - 2{\cdot}75x - 1{\cdot}2 \equiv 3(x - 1{\cdot}25)(x + 0{\cdot}32)$

to two decimal places.

SAQ1-4-1 solution

a.

$$
\begin{array}{r l}
 & \underline{\ 2x^3 - 3x^2 + 5x - 4\qquad\qquad} \\
2x + 1) & 4x^4 - 4x^3 + 7x^2 - 3x - 4 \\
 & \underline{4x^4 + 2x^3} \\
 & \quad -6x^3 + 7x^2 - 3x - 4 \\
 & \quad \underline{-6x^3 - 3x^2} \\
 & \qquad 10x^2 - 3x - 4 \\
 & \qquad \underline{10x^2 + 5x} \\
 & \qquad\qquad -8x - 4 \\
 & \qquad\qquad \underline{-8x - 4} \\
 & \qquad\qquad\quad 0
\end{array}
$$

$\therefore\ (4x^4 - 4x^3 + 7x^2 - 3x - 4) \div (2x + 1) \equiv 2x^3 - 3x^2 + 5x - 4$

b.

$$
\begin{array}{r l}
 & \underline{\ x^2 + x - 1\qquad\qquad} \\
3x^2 - x + 2\ ) & 3x^4 + 2x^3 - 2x^2 - x + 6 \\
 & \underline{3x^4 - x^3 + 2x^2} \\
 & \quad 3x^3 - 4x^2 - x + 6 \\
 & \quad \underline{3x^3 - x^2 + 2x} \\
 & \qquad -3x^2 - 3x + 6 \\
 & \qquad \underline{-3x^2 + x - 2} \\
 & \qquad\qquad -4x + 8
\end{array}
$$

Quotient is $x^2 + x - 1$ with a remainder of $-4x + 8$.

$\therefore\ (3x^4 + 2x^3 - 2x^2 - x + 6) \div (3x^2 - x + 2)$

$$\equiv\ x^2 + x - 1 - \frac{4x - 8}{3x^2 - x + 2}$$

SAQ1-4-1 solution

c.

$$\begin{array}{r} 2 \qquad\qquad\qquad \\ x^3 + x^2 - 1 \;\overline{)\; 2x^3 + \;\; 5x \;\; - 4} \\ \underline{2x^3 + \;2x^2 - 2} \qquad \\ \underline{\underline{-\;\; 2x^2 + 5x - 2}} \end{array}$$

The Quotient is 2 with a remainder of $-2x^2 + 5x - 2$

$$\therefore\ (2x^3 + 5x - 4) \div (x^3 + x^2 - 1) \equiv 2 + \frac{-2x^2 + 5x - 2}{x^3 + x^2 - 1}$$

$$\equiv 2 - \frac{2x^2 - 5x + 2}{x^3 + x^2 - 1}$$

d.

$$\begin{array}{r} x + 5 \qquad\qquad \\ x^2 - 2x - 3 \;\overline{)\; x^3 + 3x^2 + 4x - 2} \\ \underline{x^3 - 2x^2 - 3x} \qquad \\ 5x^2 + \;\; 7x - 2 \\ \underline{5x^2 - 10x - 15} \\ \underline{\underline{17x + 13}} \end{array}$$

The Quotient is $x + 5$ with a remainder of $17x + 13$

$$\therefore\ \frac{x^3 + 3x^2 + 4x - 2}{(x+1)(x-3)} \equiv x + 5 + \frac{17x + 13}{(x+1)(x-3)}$$

SAQ1-4-1 solution

e.

$$
\begin{array}{r|l}
 & 2x^2 - x + 2 \\
\hline
3x + 7 & 6x^3 + 11x^2 - x + 11 \\
 & \underline{6x^3 + 14x^2} \\
 & \quad - 3x^2 - x + 11 \\
 & \quad - \underline{3x^2 - 7} \\
 & \qquad 6x + 11 \\
 & \qquad \underline{6x + 14} \\
 & \qquad\quad \underline{\underline{-3}}
\end{array}
$$

The Quotient is $2x^2 - x + 2$ with a remainder of –3.

Hence, $(6x^3 + 11x^2 - x + 11) \div (3x + 7) \equiv 2x^2 - x + 2 - \dfrac{3}{3x+7}$

SAQ 1-5-1
solution

$$2x + 4y = 5 \qquad ①$$
$$3x - 5y = 2 \qquad ②$$

From equation ①, $x = \dfrac{5-4y}{2}$ ③

Substituting in equation ②

$$3\left(\frac{5-4y}{2}\right) - 5y = 2$$

Multiply both sides by 2

$$3(5-4y) - 10y = 4$$
$$15 - 12y - 10y = 4$$
$$11 = 22y$$
$$y = ½ = 0 \cdot 5$$

Substitute for y in ③

$$x = \frac{5 - 4 \times ½}{2}$$
$$= \tfrac{3}{2} = 1 \cdot 5$$

Solutions : $x = 1·5$, $y = 0·5$.

SAQ 1-5-2
solution

$$1·5\, I_1 - 1·2\, I_2 = 1·8 \qquad ①$$
$$2·8\, I_1 - 8·4\, I_2 = -5·88 \qquad ②$$

From equation ① $I_1 = \dfrac{1 \cdot 2 I_1 + 1 \cdot 8}{1 \cdot 5}$ ③

Substituting in equation ② : $2 \cdot 8\left(\dfrac{1 \cdot 2 I_2 + 1 \cdot 8}{1 \cdot 5}\right) - 8 \cdot 4 I_2 = -5 \cdot 88$

Multiply by 1·5: $2 \cdot 8(1 \cdot 2 I_2 + 1 \cdot 8) - 12 \cdot 6 I_2 = -1 \cdot 68$

$$3 \cdot 36 I_2 + 5 \cdot 04 - 12 \cdot 6 I_2 = -8 \cdot 82$$
$$9 \cdot 24 I = 13 \cdot 86$$

$$I_2 = 1·5$$

Substitute in ③ :

$$I_1 = \frac{1 \cdot 2 \times 1 \cdot 5 + 1 \cdot 8}{1 \cdot 5}$$
$$= 2 \cdot 4$$

Solutions : $I_1 = 2·4$, $I_2 = 1·5$.

SAQ 1-5-3 solution

$$5x + 3y = 81 \quad ①$$
$$4x - 2y = 34 \quad ②$$

Multiply equation ① by 2 and equation ② by 3.

①×2 : $10x + 6y = 162$ ③

②×3 : $12x - 6y = 102$ ④

Add equations ③ and ④

③+④ : $22x = 264$

$\therefore x = 12$

Back-substitute in equation ②

$$48 - 2y = 34$$
$$2y = 14$$
$$\therefore y = 7$$

Solutions : $x = 12$, $y = 7$.

SAQ 1-5-4 solution

$$2{\cdot}5\,I_1 + 1{\cdot}2\,I_2 = 7{\cdot}56 \quad ①$$
$$3{\cdot}2\,I_1 - 0{\cdot}7\,I_2 = 6{\cdot}77 \quad ②$$

Multiply equation ① by 0·7 and equation ② by 1·2.

①× 0·7 : $1{\cdot}75\,I_1 + 0{\cdot}84\,I_2 = 5{\cdot}292$ ③

②×1·2 : $3{\cdot}84\,I_1 - 0{\cdot}84\,I_2 = 8{\cdot}124$ ④

Add equations ③ and ④

③+④ : $5{\cdot}59\,I_1 = 13{\cdot}416$

$\therefore I_1 = 2{\cdot}4$

Back-substitute in equation ① :

$$2{\cdot}5 \times 2{\cdot}4 + 1{\cdot}2\,I_2 = 7{\cdot}56$$
$$1{\cdot}2\,I_2 = 1{\cdot}56$$
$$\therefore I_2 = 1{\cdot}3$$

Solutions : $I_1 = 2{\cdot}4$, $I_2 = 1{\cdot}3$.

SAQ 1-5-5 solution

$$x + y - z = 6 \quad ①$$
$$3x + 2y + 4z = 1 \quad ②$$
$$x - y + 2z = -6 \quad ③$$

①+③ : $2x + z = 0$ ④

③×2 : $2x - 2y + 4z = -12$ ⑤

$3x + 2y + 4z = 1$ ②

⑤+② : $5x + 8z = -11$ ⑥

④×8 : $16x + 8z = 0$ ⑦

⑦–⑥ : $11x = 11$

$\therefore x = 1$

Back-substituting into equation ④ : $2 + z = 0$

$\therefore z = -2$

Back-substituting into equation ① : $1 + y + 2 = 6$

$\therefore y = 3$

Solutions : $x = 1$, $y = 3$, $z = -2$.

SAQ 1-5-6 solution

$$3I_1 + 2I_2 + 6I_3 = 27{\cdot}5$$
$$3I_1 + 4(I_2 - I_3) = 2{\cdot}5$$
$$6(I_3 - I_1) + 4I_2 = 19$$

Rearranging

$$3I_1 + 2I_2 + 6I_3 = 27{\cdot}5 \quad ①$$
$$3I_1 + 4I_2 - 4I_3 = 2{\cdot}5 \quad ②$$
$$-6I_1 + 4I_2 + 6I_3 = 19 \quad ③$$

①–② : $-2I_2 + 10I_3 = 25$ ④

②×2 : $6I_1 + 8I_2 - 8I_3 = 5$ ⑤

$-6I_1 + 4I_2 + 6I_3 = 19$ ③

⑤+③ : $12I_2 - 2I_3 = 24$ ⑥

⑥×5 : $60I_2 - 10I_3 = 120$ ⑦

$-2I_2 + 10I_3 = 25$ ④

⑦+④ : $58I_2 = 145$

$\therefore I_2 = 2{\cdot}5$

Back-substituting in ④ : $-5 + 10I_3 = 25$

$\therefore I_3 = 3$

Back-substituting in ②: $3I_1 + 10 - 12 = 2{\cdot}5$

$\therefore I_1 = 1{\cdot}5$

Solutions : $I_1 = 1{\cdot}5$, $I_2 = 2{\cdot}5$, $I_3 = 3$.